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Spatially dependent quantum interference effects in the detection probability of charged leptons produced in neutrino interactions or weak decay processes

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Abstract. Feynman's path amplitude formulation of quantum mechanics is used to analyse the production of charged leptons from charged current weak interaction processes. For neutrino induced reactions the interference effects predicted are usually called "neutrino oscillations". Similar effects in the detection of muons from pion decay are here termed "muon oscillations". Processes considered include pion decay (at rest and in flight), and muon decay and nuclear β-decay at rest. In all cases studied, a neutrino oscillation phase different from the conventionally used one is found.

1 Introduction

The quantum mechanical description of neutrino oscillations [1,2] has been the subject of much discussion and debate in the recent literature. The "standard" oscillation formula [3], yielding an oscillation phase[1], at distance L from the neutrino source, between neutrinos, of mass m_1 and m_2 and momentum P, of[2]

$$\phi_{12} = \frac{(m_1^2 - m_2^2)L}{2P}, \tag{1.1}$$

is derived on the assumption of equal momentum and equal production times of the two neutrino mass eigenstates. Other authors have proposed, instead, equal energies [4] or velocities [5] at production, confirming, in both cases, the result of the standard formula. The latter reference claims, however, that the standard expression for ϕ_{12} should be multiplied by a factor of two in the case of the equal energy or equal momentum hypotheses when different production times are allowed for the two mass eigenstates. However, the equal momentum, energy or velocity assumptions are all incompatible with energy-momentum conservation in the neutrino production process [6].

The present paper calculates the probabilities of oscillation of neutrinos and muons produced by pions decaying both at rest and in flight, as well as the probabilities of neutrino oscillation following muon decay or β-decay of a nucleus at rest. The calculations, which are fully covariant, are based on Feynman's reformulation of quantum mechanics [7] in terms of interfering amplitudes associated with classical space-time particle trajectories. The essential interpretational formula of this approach[3], though motivated by the seminal paper of Dirac on the Lagrangian formulation of quantum mechanics [8], and much developed later in the work of Feynman and other authors [9], was actually already given by Heisenberg in 1930[4] [10]. The application of the path amplitude formalism to neutrino or muon oscillations is particularly straightforward, since, in the covariant formulation of quantum mechanics, energy and momentum are exactly conserved at all vertices and due to the macroscopic propagation distances of the neutrinos and muons all these particles follow essentially classical trajectories (i.e. corresponding to the minima of the classical action) which are rectilinear paths with constant velocities. The essential formula of Feynman's version of quantum mechanics, to be employed in the calculations presented below, is [7,10]

$$P_{fi} = \left| \sum_{k_1} \sum_{k_2} ... \sum_{k_n} \langle f|k_1\rangle \langle k_1|k_2\rangle ... \langle k_n|i\rangle \right|^2, \tag{1.2}$$

where P_{fi} is the probability to observe a final state f, given an initial state i, and $k_j,\ j = 1, n$ are (unobserved) intermediate quantum states. In the applications to be described in this paper, which, for simplicity, are limited to the case of the first two generations of leptons, (1.2)

[1] The interference term is proportional to $\cos\phi_{12}$ or $\sin^2 \frac{\phi_{12}}{2}$.

[2] Units with $\hbar = c = 1$ are used throughout.

[3] Postulate 1 and (7) of [7].

[4] Heisenberg remarked that the fundamental formula (1.2) must be distinguished from that where the summation over intermediate states is made at the level of probabilities, rather than amplitudes, and that the distinction between the two formulae is "the centre of the whole quantum theory".

specialises to[5]

$$P_{e^-\pi^+} = \left| \sum_{k=1,2} \langle e^-|\nu_k\rangle\langle\nu_k, x_D|\nu_k, x_k\rangle\langle\nu_k|\pi^+\rangle \times \langle\pi^+, x_k|\pi^+, x_0\rangle\langle\pi^+, x_0|S_\pi\rangle \right|^2 \quad (1.3)$$

for the case of neutrino oscillations and

$$P_{e^+\pi^+} = \left| \sum_{k=1,2} \langle e^+|\mu_k^+\rangle\langle\mu_k^+, x_D|\mu_k^+, x_k\rangle\langle\mu_k^+|\pi^+\rangle \times \langle\pi^+, x_k|\pi^+, x_0\rangle\langle\pi^+, x_0|S_\pi\rangle \right|^2 \quad (1.4)$$

for the case of muon oscillations. $P_{e^-\pi^+}$ is the probability to observe the charged current neutrino interaction: $(\nu_1, \nu_2)n \to e^-p$ following the decay $\pi^+ \to \mu^+(\nu_1, \nu_2)$, while $P_{e^+\pi^+}$ is the probability to observe the decay $\mu^+ \to e^+(\nu_1, \nu_2)(\overline{\nu}_1, \overline{\nu}_2)$, after the same decay process. In (1.3) and (1.4) $|\nu_k\rangle, k = 1,2$ are neutrino mass eigenstates while $|\mu_k^+\rangle, k = 1,2$ are the corresponding recoil muon states from pion decay. $\langle\nu_k|\pi^+\rangle$, $\langle\mu_k^+|\pi^+\rangle$ and $\langle e^+|\mu_k^+\rangle$ denote invariant decay amplitudes, $\langle e^-|\nu_k\rangle$ is the invariant amplitude of the charged current neutrino interaction, $\langle p, x_2|p, x_1\rangle$ is the invariant space-time propagator of particle $p = \nu, \mu, \pi$ between the space-time points x_1 and x_2 and $\langle\pi^+, x_0|S_\pi\rangle$ is an invariant amplitude describing the production of the π^+ by the source S_π and its space-time propagation to the space-time point x_0. An important feature of the amplitudes appearing in (1.3) and (1.4) is that they are completely defined in terms of the physical neutrino mass eigenstate wavefunctions $|\nu_k\rangle$. This point will be further discussed in Sect. 5 below.

The difference of the approach used in the present paper to previous calculations presented in the literature can be seen immediately on inspection of (1.3) and (1.4). The initial state[6] is a pion at space-time point x_0, the final state an e^- or e^+ produced at space-time point x_D. These are unique points, for any given event and do not depend in any way on the masses of the unobserved neutrino eigenstates propagating from x_k to x_D in (1.3). On the other hand the (unobserved) space-time points x_k at which the neutrinos and muons are produced *do* depend on k. Indeed, because of the different velocities of the propagating neutrino eigenstates, only in this case can both neutrinos and muons (representing *alternative* classical histories of the decaying pion) both arrive simultaneously at the unique point x_D where the neutrino interaction occurs (1.3) or the muon decays (1.4).

The crucial point in the above discussion is that the decaying pion, via the different path amplitudes in (1.3) and (1.4), *interferes with itself*. To modify very slightly Dirac's famous statement[7]: "Each pion then interferes only with itself. Interference between two different pions never occurs".

Because of the different possible decay times of the pion in the two interfering path amplitudes, the pion propagators $\langle\pi^+, x_k|\pi^+, x_0\rangle$ in (1.3) and (1.4) above give important contributions to the interference phase. To the author's best knowledge, this effect has not been taken into account in any previously published calculation of neutrino oscillations.

The results found for the oscillation phase are, for pion decays at rest

$$\phi_{12}^{\nu,\pi} = \phi_{12}^{\mu,\pi} = \frac{2m_\pi m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)^2} \quad (1.5)$$

and for pion decays in flight

$$\phi_{12}^{\nu,\pi} = \frac{m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)E_\nu \cos\theta_\nu}, \quad (1.6)$$

$$\phi_{12}^{\mu,\pi} = \frac{2m_\mu^2 \Delta m^2 (m_\mu^2 E_\pi - m_\pi^2 E_\mu)L}{(m_\pi^2 - m_\mu^2)^2 E_\mu^2 \cos\theta_\mu}, \quad (1.7)$$

where

$$\Delta m^2 \equiv m_1^2 - m_2^2.$$

The superscripts indicate the particles whose propagators contribute to the interference phase. Also E_π, E_ν and E_μ are the energies of the parent π and the decay ν and μ and θ_ν, θ_μ the angles between the pion and the neutrino, muon flight directions. In (1.5) to (1.7) terms of order m_1^4, m_2^4 and higher are neglected, and in (1.6) and (1.7) ultra-relativistic kinematics with $E_{\pi,\mu} \gg m_{\pi,\mu}$ is assumed. Formulae for the oscillation phase of neutrino oscillations following muon decays or nuclear β decays at rest, calculated in a similar manner to (1.5), are given in Sect. 3 below.

A brief comment is now made on the generality and the covariant nature of the calculations presented in this paper. Although the fundamental formula (1.2) is valid in both relativistic and non-relativistic quantum mechanics, it was developed in detail by Feynman [7,9] only for the non-relativistic case. For the conditions of the calculations performed in the present paper (propagation of particles in free space) the invariant space-time propagator can either be derived (for fermions) from the Dirac equation, as originally done by Feynman [12] or, more generally, from the covariant Feynman path integral for an arbitary massive particle, as recently done in [13]. The analytical form of the propagator is

$$K(x_f, x_i; m) \simeq \left(\frac{m^2}{4\pi i s}\right) H_1^{(2)}(ims), \quad (1.8)$$

where[8]

$$s = \sqrt{(x_f - x_i)^2}$$

[5] In (1.3) and (1.4) an additional summation over unobserved states, with different physical masses of the decay muon, is omitted for simplicity. See (2.1) and (2.35) below.

[6] The pion production and propagation amplitude $\langle\pi^+, x_0|S_\pi\rangle$ contributes only a multiplicative constant to the transition probabilites. The initial state can then just as well be defined as "pion at x_0", rather than $|S_\pi\rangle$. This is done in the calculations presented in Sect. 2 below.

[7] "Each photon then interferes only with itself. Interference between two different photons never occurs" [11].

[8] The metric for four-vector products is time-like.

and $H_1^{(2)}$ is a first order Hankel function of the second kind, in agreement with [12].

In the asymptotic region where $s \gg m^{-1}$, or for the propagation of on-shell particles [13], the Hankel function reduces to an exponential and yields the configuration space propagator $\simeq \exp(-\mathrm{i}ms)$ of (2.11) below. It is also shown in [13] that energy and momentum is exactly conserved in the interactions and decays of all such "asymptotically propagating" particles. The use of quasi-classical particle trajectories and the requirement of exact energy-momentum conservation are crucial ingredients of the calculations presented below.

The structure of the paper is as follows. In the following section the case of neutrino or muon oscillations following pion decay at rest is treated. Full account is taken of the momentum wave packets of the propagating neutrinos and muons resulting from the Breit–Wigner amplitudes describing the distributions of the physical masses of the decaying pion and daughter muon. The corresponding oscillation damping corrections and phase shifts are found to be very small, indicating that the quasi-classical (constant velocity) approximation used to describe the neutrino and muon trajectories is a very good one. The incoherent effects of random thermal motion of the source pion and of finite source and detector sizes on the oscillation probabilities and the oscillation phases are also calculated. These corrections are found to be small in typical experiments, but much larger than those generated by the coherent momentum wave packets. In Sect. 3, formulae are derived to describe neutrino oscillations following muon decay at rest or the β-decay of radioactive nuclei. These are written down by direct analogy with those derived in the previous section for pion decay at rest. In Sect. 4, the case of neutrino and muon oscillations following pion decay in flight is treated. In this case the two-dimensional spatial geometry of the particle trajectories must be related to the decay kinematics of the production process. Due to the non-applicability of the ultra-relativistic approximation to the kinematics of the muon in the pion rest frame, the calculation, although straightforward, is rather tedious and lengthy for the case of muon oscillations, so the details are relegated to an appendix. Finally, in Sect. 5, previous treatments in the literature of the quantum mechanics of neutrino and muon oscillations are compared with the method and results of the present paper, and the application of the Feynman path amplitude method to heavy quark flavour oscillations and atomic physics experiments is briefly mentioned.

2 Neutrino and muon oscillations following pion decay at rest

To understand clearly the different physical hypotheses and approximations underlying the calculation of the particle oscillation effects it is convenient to analyse a precise experiment. This ideal experiment is, however, very similar to LNSD [14] and KARMEN [15] except that neutrinos are produced from pion, rather than muon, decay at rest.

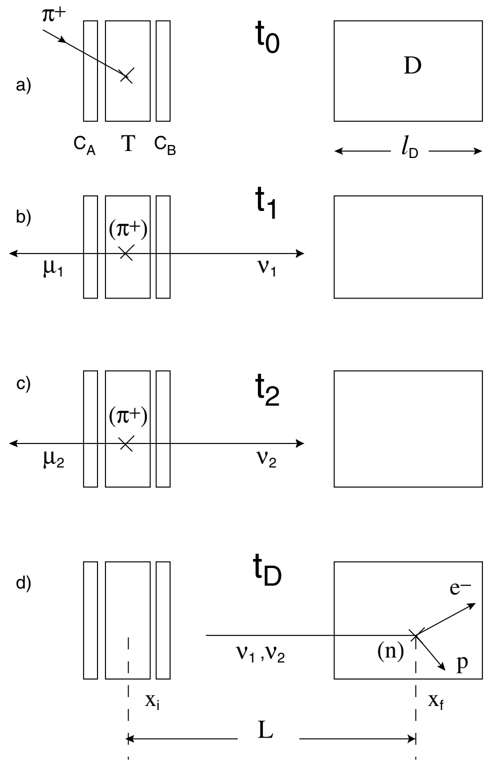

Fig. 1. The space-time description of an experiment in which neutrinos produced in the processes $\pi^+ \to \mu^+(\nu_1, \nu_2)$ are detected at distance L, via the processes $(\nu_1, \nu_2)n \to e^- p$. In **a** a π^+ comes to rest in the stopping target T at time t_0. The pion, at rest at time t_0, constitutes the initial state for the path amplitudes. In **b** and **c** are shown two alternative classical histories for the π^+; in **b**, [**c**] the pion decays into the mass eigenstate $|\nu_1\rangle$, [$|\nu_2\rangle$] at times t_1, [t_2]. If $m_1 > m_2$, and for suitable values of t_1 and t_2 ($t_2 > t_1$), the two classical histories may correspond to a common final state, shown in (d) where the neutrino interaction $(\nu_1, \nu_2)n \to e^- p$ occurs at time t_D. As the initial and final states of the two classical histories are the same, the corresponding path amplitudes must be added coherently, as in (1.2), to calculate the probability of the whole process

The different space-time events that must be considered in order to construct the probability amplitudes for the case of neutrino oscillations following pion decay at rest are shown in Fig. 1. A π^+ passes through the counter C_A, where the time t_0 is recorded, and comes to rest in a thin stopping target T (Fig. 1a). For simplicity, the case of only two neutrino mass eigenstates ν_1 and ν_2 of masses m_1 and m_2 ($m_1 > m_2$) is considered. The pion at rest constitutes the initial state of the quantum mechanical probability amplitudes. The final state is an $e^- p$ system produced, at time t_D, via the process $(\nu_1, \nu_2)n \to e^- p$ at a distance L from the decaying π^+ (Fig. 1d). Two different physical processes may produce the observed $e^- p$ final state, as shown in Fig. 1b,c, where the pion decays either at time t_1 into ν_1 or at time t_2 into ν_2. The probability amplitudes for these processes are, up to an arbitary normalisation constant, and neglecting solid angle factors in the propagators:

$$
\begin{aligned}
A_i = \int & \langle e^- p|T_R|n\nu_i\rangle U_{ei} D(x_f - x_i, t_D - t_i, m_i) \\
& \times BW(W_{\mu(i)}, m_\mu, \Gamma_\mu) \\
& \times U_{i\mu}\langle \nu_i \mu^+|T_R|\pi^+\rangle \mathrm{e}^{-\frac{\Gamma_\pi}{2}(t_i - t_0)} D(0, t_i - t_0, m_\pi) \\
& \times BW(W_\pi, m_\pi, \Gamma_\pi)\mathrm{d}W_{\mu(i)}, \qquad i = 1,2. \qquad (2.1)
\end{aligned}
$$

Note that following the conventional "fi" (final, initial) ordering of the indices of matrix elements in quantum mechanics, the path amplitude is written from right to left in order of increasing time. This ensures also correct matching of "bra" and "ket" symbols in the amplitudes. In (2.1), $\langle e^- p|T_R|n\nu_i\rangle$, $\langle \nu_i\mu^+|T_R|\pi^+\rangle$ are "reduced" invariant amplitudes of the ν charged current scattering and pion decay processes, respectively, $BW(W_{\mu(i)}, m_\mu, \Gamma_\mu)$ and $BW(W_\pi, m_\pi, \Gamma_\pi)$ are relativistic Breit–Wigner amplitudes, U_{ei} and $U_{i\mu}$ are elements of the unitary Maki–Nagagawa–Sakata (MNS) matrix [16], $U_{\alpha i}$, describing the charged current coupling of a charged lepton, α ($\alpha = e, \mu, \tau$), to the neutrino mass eigenstate i. The reduced invariant amplitudes are defined by factoring out the MNS matrix element from the amplitude for the process. For example: $\langle e^- p|T|n\nu_i\rangle = U_{ei}\langle e^- p|T_R|n\nu_i\rangle$. Since the purely kinematical effects of the non-vanishing neutrino masses are expected to be very small, the reduced matrix elements may be assumed to be lepton flavour independent: $\langle e^- p|T_R|n\nu_i\rangle \simeq \langle e^- p|T_R|n\nu_0\rangle$ where ν_0 denotes a massless neutrino. In (2.1), D is the Lorentz-invariant configuration space propagator [13,12] of the pion or neutrino. The pole masses and total decay widths of the pion and muon are denoted by m_π, Γ_π and m_μ, Γ_μ respectively. For simplicity, phase space factors accounting for different observed final states are omitted in (2.1) and subsequent formulae.

Because the amplitudes and propagators in (2.1) are calculated using relativistic quantum field theory, and the neutrinos propagate over macroscopic distances, it is a good approximation, as already discussed in the previous section, to assume that there is exact energy-momentum conservation in the pion decay process and that the neutrinos are on their mass shells, i.e. $p_i^2 = m_i^2$, where p_i is the neutrino energy-momentum four-vector. In these circumstances the neutrino propagators correspond to classical, rectilinear, particle trajectories.

The pion and muon are unstable particles whose physical masses W_π and $W_{\mu(i)}$ differ from the pole masses m_π and m_μ appearing in the Breit–Wigner amplitudes and covariant space-time propagators in (2.1). The neutrino momentum P_i will depend on these physical masses according to the relation

$$
P_i = \frac{\left[[W_\pi^2 - (m_i + W_{\mu(i)})^2][W_\pi^2 - (m_i - W_{\mu(i)})^2]\right]^{\frac{1}{2}}}{2W_\pi} \qquad (2.2)
$$

Note that, because the initial state pion is the same in the two path amplitudes in (2.1) W_π does not depend on the neutrino mass index i. However, since the pion decays resulting in the production of ν_1 and ν_2 are independent physical processes, the physical masses of the unobserved muons, $W_{\mu(i)}$, $i = 1,2$, recoiling against the two neutrino mass eigenstates are not, in general, the same. In the following kinematical calculations sufficient accuracy is achieved by retaining only quadratic terms in the neutrino masses and terms linear in the small quantities $\delta_\pi = W_\pi - m_\pi$, $\delta_i = W_{\mu(i)} - m_\mu$. This allows simplification of the relativistic Breit–Wigner amplitudes:

$$
\begin{aligned}
BW(W, m, \Gamma) &\equiv \frac{\Gamma m}{W^2 - m^2 + \mathrm{i}m\Gamma} \\
&= \frac{\Gamma m}{\delta(2m + \delta) + \mathrm{i}m\Gamma} \\
&= \frac{\Gamma}{2\left(\delta + \mathrm{i}\frac{\Gamma}{2}\right)} + O(\delta^2) \\
&\equiv BW(\delta, \Gamma) + O(\delta^2). \qquad (2.3)
\end{aligned}
$$

Developing (2.2) up to first order in m_i^2, δ_i and δ_π yields the relation

$$
\begin{aligned}
P_i = P_0 \Bigg[1 &- \frac{m_i^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} + \frac{\delta_\pi}{m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)} \\
&- \frac{2\delta_i m_\mu}{m_\pi^2 - m_\mu^2} + \frac{\delta_\pi m_i^2(m_\pi^2 + m_\mu^2)}{m_\pi(m_\pi^2 - m_\mu^2)^2}\Bigg], \qquad (2.4)
\end{aligned}
$$

where

$$
P_0 = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.8\,\mathrm{MeV}. \qquad (2.5)
$$

The term $\simeq \delta_\pi m_i^2$ which is also included in (2.4) gives a negligible $\mathrm{O}(m_i^4)$ contribution to the neutrino oscillation formula. For muon oscillations, however, it gives a term of $\mathrm{O}(m_i^2)$ in the interference term, as discussed below. Similarly, the exact formula for the neutrino energy

$$
E_i = \frac{W_\pi^2 - W_{\mu(i)}^2 + m_i^2}{2W_\pi} \qquad (2.6)
$$

in combination with (2.4) gives for the neutrino velocity

$$
\begin{aligned}
v_i = \frac{P_i}{E_i} = 1 - \frac{m_i^2}{2P_0^2}&\left[1 - \frac{2\delta_\pi(m_\pi^2 + m_\mu^2)}{m_\pi(m_\pi^2 - m_\mu^2)} + \frac{4\delta_i m_\mu}{m_\pi^2 - m_\mu^2}\right] \\
&+ O(m_i^4, \delta_\pi^2, \delta_i^2). \qquad (2.7)
\end{aligned}
$$

This formula will be used below to calculate the neutrino times-of-flight t_i^{fl}, $i = 1,2$.

From the unitarity of the MNS matrix, the elements $U_{\alpha i}$ may be expressed in terms of a single real angular parameter θ:

$$
U_{e1} = U_{1e} = U_{\mu 2} = U_{2\mu} = \cos\theta, \qquad (2.8)
$$

$$
U_{e2} = U_{2e} = -U_{\mu 1} = -U_{1\mu} = \sin\theta. \qquad (2.9)
$$

The parts of the amplitudes requiring the most careful discussion are the invariant space-time propagators D, as it is mainly their treatment that leads to the different result for the neutrino oscillation phase found in the present paper, as compared to those having previously appeared in the literature. In the limit of large time-like separations, the propagator may be written as [13,12]

$$D(\Delta x, \Delta t, m) = \left(\frac{m}{2\pi \mathrm{i}\sqrt{(\Delta t)^2 - (\Delta x)^2}} \right)^{\frac{3}{2}} \times \exp\left[-\mathrm{i}m\sqrt{(\Delta t)^2 - (\Delta x)^2} \right]. \quad (2.10)$$

D is the amplitude for a particle, originally at a space-time point $(\mathbf{x}_i, t_i)$, to be found at $(\mathbf{x}_f, t_f)$ and $\Delta\mathbf{x} \equiv \mathbf{x}_f - \mathbf{x}_i$, $\Delta t \equiv t_f - t_i$. In the following, according to the geometry of the experiment shown in Fig. 1, only one spatial coordinate will be considered ($\Delta x \equiv x_f - x_i$) and only the exponential factor in (2.10), containing the essential phase information for particle oscillations will be retained in the amplitudes. Solid angle correction factors, taken into account by the factor in large brackets in (2.10), are here neglected, but are easily included in the final oscillation formulae. Writing then

$$\begin{aligned} D(\Delta x, \Delta t, m) &\simeq \exp\left[-\mathrm{i}m\sqrt{(\Delta t)^2 - (\Delta x)^2} \right] \\ &= \exp\left[-\mathrm{i}m\Delta\tau \right] \\ &\equiv \exp[-\mathrm{i}\Delta\phi], \end{aligned} \quad (2.11)$$

it can be seen that the increment in phase of the propagator, $\Delta\phi$, when the particle undergoes the space-time displacement (Δx, Δt) is a Lorentz-invariant quantity equal to the product of the particle mass and the increment, $\Delta\tau$, of proper time. Using the relativistic time dilatation formula

$$\Delta t = \gamma \Delta\tau = \frac{E}{m}\Delta\tau, \quad (2.12)$$

and also the relation, corresponding to a classical, rectilinear, particle trajectory,

$$\Delta t = \frac{L}{v} = \frac{E}{p}L, \quad (2.13)$$

gives, for the phase increments corresponding to the paths of the neutrinos and the pion in Fig. 1

$$\begin{aligned} \Delta\phi_i^\nu &= m_i \Delta\tau_i = \frac{m_i^2}{E_i}\Delta t_i = \frac{m_i^2}{P_i}L \\ &= \frac{m_i^2 L}{P_0}\left[1 - \frac{\delta_\pi}{m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)} + \frac{2\delta_i m_\mu}{m_\pi^2 - m_\mu^2} \right], \quad (2.14) \\ \Delta\phi_i^\pi &= m_\pi(t_i - t_0) = m_\pi(t_D - t_0) - \frac{m_\pi L}{v_i} \\ &= m_\pi(t_D - t_0) \\ &\quad - m_\pi L \left\{ 1 + \frac{m_i^2}{2P_0^2}\left[1 - \frac{2\delta_\pi}{m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)} \right.\right. \\ &\quad \left.\left. + \frac{4\delta_i m_\mu}{m_\pi^2 - m_\mu^2} \right] \right\}, \quad (2.15) \end{aligned}$$

where terms of $\mathrm{O}(m_i^4)$ and higher are neglected.

Making the substitution $t_i - t_0 \to t_D - t_0 - L/v_i$ in the exponential damping factor due to the pion lifetime in (2.1) and using (2.3), (2.11), (2.14) and (2.15), (2.1) may be written as

$$\begin{aligned} A_i = \int &\langle e^- p|T_R|n\nu_0\rangle U_{ei}U_{i\mu}\langle\nu_0\mu^+|T_R|\pi^+\rangle \\ &\times \frac{\Gamma_\mu}{2}\frac{\mathrm{e}^{\mathrm{i}\alpha_i\delta_i}}{\left(\delta_i + \mathrm{i}\frac{\Gamma_\mu}{2}\right)} \\ &\times \frac{\Gamma_\pi}{2}\frac{\mathrm{e}^{\mathrm{i}\alpha_\pi(i)\delta_\pi}}{\left(\delta_\pi + \mathrm{i}\frac{\Gamma_\pi}{2}\right)}\mathrm{e}^{\mathrm{i}\phi_0 - \frac{\Gamma_\pi}{2}(t_D - t_0 - t_i^{\mathrm{fl}})} \\ &\times \exp\mathrm{i}\left[\frac{m_i^2}{P_0}\left(\frac{m_\pi}{2P_0} - 1 \right)L \right]\mathrm{d}\delta_i, \quad i = 1, 2, \end{aligned} \quad (2.16)$$

where

$$\phi_0 \equiv m_\pi(L - t_D + t_0), \quad (2.17)$$

$$\alpha_i \equiv \frac{4m_i^2 m_\mu m_\pi(m_\pi^2 + m_\mu^2)L}{(m_\pi^2 - m_\mu^2)^3}, \quad (2.18)$$

$$\alpha_\pi(i) \equiv -\frac{2m_i^2(m_\pi^2 + m_\mu^2)^2 L}{(m_\pi^2 - m_\mu^2)^3} \quad (2.19)$$

and

$$t_i^{\mathrm{fl}} = L\left(1 + \frac{m_i^2}{2P_0^2} \right) + O(m_i^4). \quad (2.20)$$

In (2.18) and (2.19) imaginary parts of relative size $\simeq \Gamma_\pi/m_\pi \simeq 2.0 \times 10^{-16}$ are neglected.

To perform the integral over δ_i in (2.16) it is convenient to approximate the modulus squared of the Breit–Wigner amplitude by a Gaussian, via the substitution

$$\begin{aligned} \frac{\frac{\Gamma}{2}}{\delta + \mathrm{i}\frac{\Gamma}{2}} &= \frac{\Gamma}{2}\left(\frac{\delta - \mathrm{i}\frac{\Gamma}{2}}{\delta^2 + \frac{\Gamma^2}{4}} \right) \\ &\to \frac{2}{\Gamma}\left(\delta - \mathrm{i}\frac{\Gamma}{2} \right)\exp\left(-\frac{3\delta^2}{\Gamma^2} \right), \end{aligned} \quad (2.21)$$

where the width of the Gaussian is chosen so that it has approximately the same full width at half maximum, Γ, as the Breit–Wigner function. After the substitution (2.21), the integral over δ_i in (2.16) is easily evaluated by a change of variable to "complete the square" in the argument of the exponential, with the result

$$\begin{aligned} I_i &= \frac{2}{\Gamma_\mu}\int_{-\infty}^{\infty}\left(\delta_i - \mathrm{i}\frac{\Gamma_\mu}{2} \right)\exp\left(-\frac{3\delta_i^2}{\Gamma_\mu^2} + \mathrm{i}\alpha_i\delta_i \right)\mathrm{d}\delta_i \\ &= \mathrm{i}\sqrt{\frac{\pi}{3}}\Gamma_\mu \exp\left(-\frac{\alpha_i^2\Gamma_\mu^2}{12} \right)\left[\frac{\alpha_i\Gamma_\mu}{3} - 1 \right]; \end{aligned} \quad (2.22)$$

(2.18) gives, for α_i, the numerical value

$$\alpha_i = 3.1 \times 10^{14}\left(\frac{m_i}{m_\pi} \right)^2 L(m)\,\mathrm{MeV}^{-1}. \quad (2.23)$$

For the typical physically interesting values (see below) of $m_i = 1\,\mathrm{eV}$ and $L = 30\,\mathrm{m}$, α_i takes the value $0.48\,\mathrm{MeV}^{-1}$, so that

$$\alpha_i\Gamma_\mu = 0.48 \times 3.00 \times 10^{-16} = 1.4 \times 10^{-16}$$

Then, to very good accuracy, $I_1 = I_2 = -\mathrm{i}\sqrt{\pi/3}\Gamma_\mu$, independently of the neutrino mass. It follows that for neutrino oscillations, the muon mass dependence of the amplitudes may be neglected for any physically interesting values of m_i and L.

From (2.16) and (2.22) the probability to observe the reactions $(\nu_1, \nu_2)n \to e^- p$ at distance L from the pion decay point and at time t_D is

$$\begin{aligned} P(e^-p|L, t_D) &= |A_1 + A_2|^2 \\ &= \frac{\pi \Gamma_\mu^2}{3} |\langle e^- p|T_R|n\nu_0\rangle|^2 |\langle \nu_0 \mu^+|T_R|\pi^+\rangle|^2 \\ &\times \sin^2\theta \cos^2\theta \mathrm{e}^{-\Gamma_\pi(t_D - t_0)} \frac{\Gamma_\pi^2}{4\left(\delta_\pi^2 + \frac{\Gamma_\pi^2}{4}\right)} \\ &\times \left\{ \mathrm{e}^{\Gamma_\pi t_1^{\mathrm{fl}}} + \mathrm{e}^{\Gamma_\pi t_2^{\mathrm{fl}}} \right. \\ &\quad - 2\mathrm{e}^{\Gamma_\pi \frac{(t_1^{\mathrm{fl}} + t_2^{\mathrm{fl}})}{2}} \mathrm{Re} \exp \mathrm{i} \left[\frac{\Delta m^2}{P_0} \left(\frac{m_\pi}{2P_0} - 1 \right) L \right. \\ &\quad \left. \left. + [\alpha_\pi(1) - \alpha_\pi(2)]\delta_\pi \right] \right\}. \end{aligned} \tag{2.24}$$

The time-dependent exponential factors in the curly brackets of (2.24) are easily understood. If $m_1 > m_2$ then $t_1^{\mathrm{fl}} > t_2^{\mathrm{fl}}$. This implies that the neutrino of mass m_1 results from an earlier decay than the neutrino of mass m_2, in order to be detected at the same time. Because of the exponential decrease with time of the pion decay amplitude, the contribution to the probability of the squared amplitude for the neutrino of mass m_1 is larger. The interference term resulting from the product of the decay amplitudes of the two neutrinos of different mass has an exponential factor that is the harmonic mean of those of the squared amplitudes for each neutrino mass eigenstate and so is also suppressed relative to the squared amplitude for the neutrino of mass m_1. The integral over the physical pion mass is readily performed by replacing the Breit–Wigner function by a Gaussian as in (2.21). This leads to an overall multiplicative constant $\sqrt{\pi/3}\Gamma_\pi$ and a factor

$$F^\nu(W_\pi) = \exp[-(\alpha_\pi(1) - \alpha_\pi(2))^2 \Gamma_\pi^2/12] \tag{2.25}$$

multiplying the interference term. For $\Delta m^2 = (1\,\mathrm{eV})^2$ and $L = 30\,\mathrm{m}$ the numerical value of this factor is $\exp(-1.3 \times 10^{-29})$. This tiny correction is neglected in the following equations.

Integrating over t_D gives the average probability to observe the e^-p final state at distance L:

$$\begin{aligned} P(e^-p|L) &= \frac{\pi^{\frac{3}{2}} \Gamma_\mu^2}{3\sqrt{3}} |\langle e^- p|T_R|n\nu_0\rangle|^2 |\langle \nu_0 \mu^+|T_R|\pi^+\rangle|^2 \sin^2\theta \cos^2\theta \\ &\times \left\{ 1 - \exp\left[-\frac{\Gamma_\pi m_\pi^2 \Delta m^2}{(m_\pi^2 - m_\mu^2)^2} L \right] \cos \frac{2 m_\pi m_\mu^2 \Delta m^2}{(m_\pi^2 - m_\mu^2)^2} L \right\}, \end{aligned} \tag{2.26}$$

where all kinematical quantities are expressed in terms of Δm^2, m_π and m_μ. Note that the minimum value of t_D is $t_0 + t_1^{\mathrm{fl}}$, $t_0 + t_2^{\mathrm{fl}}$ and $t_0 + t_1^{\mathrm{fl}}$ for the squared amplitude terms of neutrinos of mass m_1, m_2 and the interference term, respectively. On integrating over t_D, the squared amplitude terms give equal contributions, the larger amplitude for mass m_1 being exactly compensated by a smaller range of integration. The exponential damping factor in the interference term in (2.26) is derived using the relations

$$\begin{aligned} t_1^{\mathrm{fl}} - t_2^{\mathrm{fl}} &= \left(\frac{1}{v_1(\nu)} - \frac{1}{v_2(\nu)} \right) L \\ &\simeq (v_2(\nu) - v_1(\nu))L \end{aligned} \tag{2.27}$$

and

$$v_i(\nu) = 1 - \frac{m_i^2}{2P_0^2} + O(m_i^4), \quad i = 1, 2, \tag{2.28}$$

to obtain

$$t_1^{\mathrm{fl}} - t_2^{\mathrm{fl}} = \frac{(m_1^2 - m_2^2)L}{2P_0^2} = \frac{\Delta m^2 L}{2P_0^2} \tag{2.29}$$

The damping factor arises because the difference in the times-of-flight of the two neutrino paths is limited by the pion lifetime. It will be seen below, however, that for distances L of practical interest for the observation of neutrino oscillations, the damping effect is tiny.

The part of the oscillation phase in (2.24) originating from the neutrino propagators (the term associated with the number "1" within the large curved brackets) differs by a factor two from the corresponding expression in the standard formula. The contribution to the oscillation phase of the propagator of the decaying pion (the term associated with $m_\pi/(2P_0)$ in the large curved brackets of (2.24)) has not been taken into account in any published calculation known to the author of the present paper. The oscillation phase in (2.26) is $2m_\mu^2/(m_\pi^2 - m_\mu^2) = 2.685$ times larger than that given by the standard formula (1.1). For $L = 30\,\mathrm{m}$, as in the LNSD experiment, the first oscillation maximum occurs for $\Delta m^2 = 0.46\,(\mathrm{eV})^2$. Denoting by $\phi_{12}^{\nu,\pi}$ the phase of the cosine interference term in (2.26), the pion lifetime damping factor can be written as

$$\begin{aligned} F^\nu(\Gamma_\pi) &= \exp\left(-\Gamma_\pi \frac{m_\pi}{2m_\mu^2} \phi_{12}^{\nu,\pi} \right) \\ &= \exp(-1.5810^{-16} \phi_{12}^{\nu,\pi}), \end{aligned} \tag{2.30}$$

so the damping effect is vanishingly small when $\phi_{12}^{\nu,\pi} \simeq 1$.

The oscillation formula (2.26) is calculated on the assumption that the decaying pion is at rest at the precisely defined position x_i. In fact, the positive pion does not bind with the atoms of the target, but will rather undergo random thermal motion. This has three effects: an uncertainty in the value of x_i, a Doppler shift of the neutrino energy and a time dilatation correction factor of $1/\gamma_\pi$ in (2.15) for the pion phase increment. Assuming that the target is at room temperature ($T = 270\,\mathrm{K}$), the mean kinetic energy of $3kT/2$ corresponds to a mean pion momentum of $2.6 \times 10^{-3}\,\mathrm{MeV}$ and a mean velocity of $\simeq 5.6\,\mathrm{km/s}$. The pion will move, in one mean lifetime ($2.6 \times 10^{-8}\,\mathrm{s}$), a distance of 146 μm. This is negligible as compared to L (typically ≥ 30 m) and so (2.26) requires no modification to account for this effect.

The correction factor due to the Doppler effect and time dilatation is readily calculated on the assumption of a Maxwell–Boltzmann distribution of the pion momentum:

$$\frac{\mathrm{d}N}{\mathrm{d}p_\pi} \simeq p_\pi^2 \exp\left(-\frac{p_\pi^2}{\overline{p}_\pi^2}\right). \quad (2.31)$$

Here $\overline{p}_\pi = \sqrt{2kTm_\pi} = 2.64 \times 10^{-3}$ MeV. Details of the calculation are given in Appendix A. The interference term in (2.26) is modified by a damping factor:

$$F^\nu(\mathrm{Dop}) = \exp\left\{-\left(\frac{\overline{p}_\pi \Delta m^2}{2m_\pi P_0}\left[\frac{m_\pi}{P_0} - 1\right] L\right)^2\right\}, \quad (2.32)$$

while the argument of the cosine term acquires an additional phase factor:

$$\phi^\nu(\mathrm{Dop}) = \frac{3}{4}\left(\frac{\overline{p}_\pi}{m_\pi}\right)^2 \frac{\Delta m^2}{P_0}\left[\frac{3m_\pi}{2P_0} - 1\right] L. \quad (2.33)$$

For $\phi_{12}^{\nu,\pi} = 1$, $F^\nu(\mathrm{Dop}) = 1 - 6.7 \times 10^{-10}$ and $\phi^\nu(\mathrm{Dop}) = 1.2 \times 10^{-9}$.

If the target in which the pion stops is of thickness ℓ_T, then the effect of different stopping points of the π (assumed uniformly distributed) is to multiply the interference term in (2.26) by the factor

$$F_{\mathrm{Targ}}^\nu = \frac{(m_\pi^2 - m_\mu^2)^2}{m_\pi m_\mu^2 \Delta m^2 \ell_T} \sin\left(\frac{m_\pi m_\mu^2 \Delta m^2 \ell_T}{(m_\pi^2 - m_\mu^2)^2}\right). \quad (2.34)$$

If the position of the neutrino interaction point within the target has an uncertainy of $\pm\ell_D/2$ a similar correction factor is found, with the replacement $\ell_T \to \ell_D$ in (2.34). The calculation of this correction factor is also described in Appendix A.

The ideal experiment, described above, to study neutrino oscillations, is easily adapted to the case of oscillations in the decay probability of muons produced by charged pion decay at rest. As before, the pion stops in the target T at time t_0 (see Fig. 2a). At time t_1 the pion decays into ν_1 and the corresponding recoil muon (μ_1), whose passage is recorded in the counter $\mathrm{C_B}$ (Fig. 2b). Similarly, a decay into ν_2 and μ_2 may occur at time t_2 (Fig. 2c). With a suitable choice of the times t_1 and t_2, such that muons following the alternative paths both arrive at the same time t_D at the point x_f, interference occurs between the path amplitudes when muon decay occurs at the space-time point (x_f, t_D) in the detector D (Fig. 2d). The probability for two *classical* trajectories to arrive at *exactly* the same space-time point of course vanishes. The correct way to consider the quantum mechanical calculation is rather to ask *given that* the muon decay occurs at the point (x, t_D), does the muon recoil against ν_1 or ν_2? If these two possiblities are not distinguished by the measurement of the decay process, the corresponding probability *amplitudes* (not probabilities) must be added

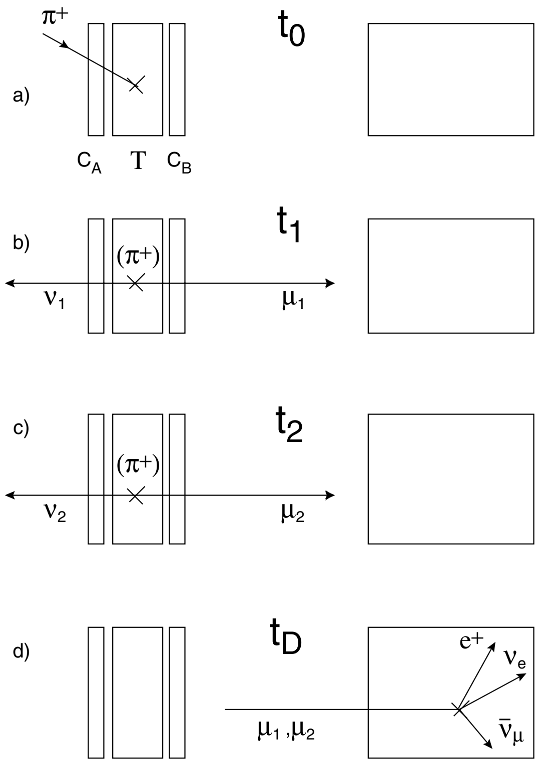

Fig. 2. The space-time description of an experiment in which muons produced in the processes $\pi^+ \to \mu^+(\nu_1, \nu_2)$ are detected at distance L, via decay processes $\mu^+ \to e^+(\nu_1, \nu_2)(\overline{\nu}_1, \overline{\nu}_2)$, denoted, conventionally, as "$\mu^+ \to e^+\nu_e\overline{\nu}_\mu$" As in Fig. 1b and c show alternative classical histories of the stopped π^+. If $m_1 > m_2$ the velocity of μ_1 is less than that of μ_2, and provided that $t_2 > t_1$, the muons may arrive at the same spatial point at the same time t_D in both classical histories. If the muons are detected at this space-time point in any way (not necessarily by the observation of muon decay as shown in **c**) interference between the corresponding path amplitudes occurs, according to (1.2), just as in the case of neutrino detection

in the calculation of the probability of the observed decay process.

The path amplitudes corresponding to muons recoiling against neutrinos of mass m_1 and m_2 are

$$\begin{aligned} A_{(kl)i}^{(\mu)} &= \int \langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle \mathrm{e}^{-\frac{\Gamma_\mu v_i^\mu}{2\gamma_i^\mu}L} D(x_f - x_i, t_D - t_i, m_\mu) \\ &\times BW(W_{\mu(i)}, m_\mu, \Gamma_\mu) U_{i\mu}\langle\nu_i\mu^+|T_R|\pi^+\rangle \\ &\times \mathrm{e}^{-\frac{\Gamma_\pi}{2}(t_i - t_0)} D(0, t_i - t_0, m_\pi) BW(W_\pi, m_\pi, \Gamma_\pi)\mathrm{d}W_{\mu(i)}, \\ &\quad i = 1, 2. \end{aligned} \quad (2.35)$$

The various factors in these equations are defined, mutatis mutandis, as in (2.1).

With the same approximations concerning the neutrino masses and the physical pion and muon masses as those made above, the velocity of the muon recoiling against the neutrino mass eigenstate ν_i is

$$v_i^\mu = v_0^\mu \left[1 - \frac{4m_i^2 m_\pi^2 m_\mu^2}{(m_\pi^2 - m_\mu^2)^2(m_\pi^2 + m_\mu^2)} + \frac{4\delta_\pi m_\pi m_\mu^2}{m_\pi^4 - m_\mu^4} - \frac{4\delta_i m_\mu m_\pi^2}{m_\pi^4 - m_\mu^4} - \frac{8\delta_\pi m_i^2 m_\pi^3 m_\mu^2(3m_\mu^2 - m_\pi^2)}{(m_\pi^4 - m_\mu^4)^2(m_\pi^2 - m_\mu^2)}\right], \tag{2.36}$$

where

$$v_0^\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}. \tag{2.37}$$

Comparing with (2.7), it can be seen that for the muon case, unlike that where neutrino interactions are observed, there are pion and muon mass dependent correction terms that are independent of the neutrino masses, implying a velocity smearing effect due to the physical pion and muon masses that is $\simeq m_\pi^2/m_i^2$ larger than for the case of neutrino oscillations.

The phase increments corresponding to the paths of the muons and the pion in Fig. 2 are, using (2.4)[9] and (2.12)–(2.15) and (2.36)

$$\begin{aligned}\Delta\phi_i^\mu &= \frac{m_\mu^2 L}{P_i^\mu} \\ &= \frac{m_\mu^2 L}{P_0}\left[1 + \frac{m_i^2 E_0^\mu}{2m_\pi P_0^2} - \frac{\delta_\pi}{m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)} + \frac{2\delta_i m_\mu}{m_\pi^2 - m_\mu^2} - \frac{\delta_\pi m_i^2}{m_\pi}\frac{(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}\right],\end{aligned} \tag{2.38}$$

$$\begin{aligned}\Delta\phi_i^{\pi(\mu)} &= m_\pi(t_i - t_0) = m_\pi(t_D - t_0) - \frac{m_\pi L}{v_i^\mu} \\ &= m_\pi(t_D - t_0) - \frac{m_\pi L}{v_0^\mu}\left[1 + \frac{4m_i^2 m_\pi^2 m_\mu^2}{(m_\pi^2 - m_\mu^2)^2(m_\pi^2 + m_\mu^2)} - \frac{4\delta_\pi m_\pi m_\mu^2}{m_\pi^4 - m_\mu^4} + \frac{4\delta_i m_\mu m_\pi^2}{m_\pi^4 - m_\mu^4} + \frac{8\delta_\pi m_i^2 m_\pi^3 m_\mu^2(3m_\mu^2 - m_\pi^2)}{(m_\pi^4 - m_\mu^4)^2(m_\pi^2 - m_\mu^2)}\right],\end{aligned} \tag{2.39}$$

where

$$E_0^\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}. \tag{2.40}$$

Using (2.11), (2.38) and (2.39) to re-write the space-time propagators in (2.35), as well as (2.3) for the Breit–Wigner amplitudes, gives

$$\begin{aligned}A_{(kl)i}^{(\mu)} = &\int \langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle e^{-\frac{\Gamma_\mu v_0^\mu L}{2\gamma_0^\mu}} U_{i\mu}\langle \nu_i\mu^+|T|\pi^+\rangle \\ &\times \frac{\Gamma_\mu}{2}\frac{e^{i\alpha^\mu\delta_i}}{\left(\delta_i + i\frac{\Gamma_\mu}{2}\right)}\frac{\Gamma_\pi}{2}\frac{e^{i\alpha_\pi^\mu(i)\delta_\pi}}{\left(\delta_\pi + i\frac{\Gamma_\pi}{2}\right)} \\ &\times e^{i\phi_0^\mu - \frac{\Gamma_\pi}{2}(t_D - t_0 - t_{\mu(i)}^{fl})} \\ &\times \exp i\left[\frac{m_\mu^2 m_i^2}{2P_0^3}\left(1 - \frac{E_0^\mu}{m_\pi}\right)L\right] d\delta_i, \\ &i = 1, 2,\end{aligned} \tag{2.41}$$

where

$$\phi_0^\mu \equiv m_\pi\left(\frac{L}{v_0^\mu} - t_D + t_0\right) - \frac{m_\mu^2 L}{P_0}, \tag{2.42}$$

$$\alpha^\mu \equiv \frac{4m_\mu m_\pi L}{m_\pi^2 - m_\mu^2}, \tag{2.43}$$

$$\alpha_\pi^\mu(i) \equiv -\frac{2m_\mu^2 L}{m_\pi^2 - m_\mu^2}\left[1 - \frac{m_i^2(5m_\pi^6 - 11m_\pi^4 m_\mu^2 - m_\pi^2 m_\mu^4 - m_\mu^6)}{(m_\pi^4 - m_\mu^4)(m_\pi^2 - m_\mu^2)^2}\right] \tag{2.44}$$

and

$$t_{\mu(i)}^{fl} = L\left(\frac{1}{v_0^\mu} + \frac{m_i^2 m_\mu^2}{2m_\pi P_0^3}\right), \tag{2.45}$$

where, as in (2.18) and (2.19), imaginary parts of order Γ_π/m_π are neglected. Making the substitution (2.21) and performing the integral over δ_i according to (2.22), the following formula is found for the probability for muon decay at distance L and time t_D:

$$\begin{aligned}P(e^+\nu_k\overline{\nu}_l|L, t_D) &= |A_{(kl)1}^{(\mu)} + A_{(kl)2}^{(\mu)}|^2 \\ &= \frac{\pi\Gamma_\mu^2}{3}e^{-\frac{(\alpha^\mu\Gamma_\mu)^2}{6}}\left[1 - \frac{\alpha^\mu\Gamma_\mu}{3}\right]^2 |\langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle|^2 e^{-\frac{\Gamma_\mu v_0^\mu}{\gamma_0^\mu}L} \\ &\quad\times |\langle \nu_0\mu^+|T_R|\pi^+\rangle|^2 e^{-\Gamma_\pi(t_D - t_0)}\frac{\Gamma_\pi^2}{4\left(\delta_\pi^2 + \frac{\Gamma_\pi^2}{4}\right)} \\ &\quad\times\Big\{\sin^2\theta e^{\Gamma_\pi t_{\mu(1)}^f} + \cos^2\theta e^{\Gamma_\pi t_{\mu(2)}^f} \\ &\quad - 2\sin\theta\cos\theta e^{\frac{\Gamma_\pi}{2}(t_{\mu(1)}^f + t_{\mu(2)}^f)} \\ &\quad\times \mathrm{Re}\exp i\Big[\frac{m_\mu^2\Delta m^2}{2P_0^3}\left(1 - \frac{E_0^\mu}{m_\pi}\right)L \\ &\quad + [\alpha_\pi^\mu(1) - \alpha_\pi^\mu(2)]\delta_\pi\Big]\Big\},\end{aligned} \tag{2.46}$$

where the effect of the non-zero neutrino masses are neglected in the reduced pion decay amplitude so that $\langle \nu_i\mu^+|T_R|\pi^+\rangle \simeq \langle \nu_0\mu^+|T_R|\pi^+\rangle$ and this amplitude is a common factor in both path amplitudes. The muon path difference yields the term associated with E_0^μ/m_π in the interference phase in (2.46) while the pion path is associated with the number "1" in the large curved brackets. The numerical value of the damping factor:

$$F^\mu(W_\mu) = \exp\left[-\frac{(\alpha^\mu\Gamma_\mu)^2}{6}\right]\left[1 - \frac{\alpha^\mu\Gamma_\mu}{3}\right]^2, \tag{2.47}$$

resulting from the integral over the physical muon mass is, for $L = 30$ m, 0.774, so, unlike for the case of neutrino

[9] Note that in the pion rest frame $P_i = P_i^\mu$.

oscillations, the correction is by no means negligible. This is because, in the muon oscillation case, the leading term of α^μ is not proportional to the neutrino mass squared. The non-leading terms proportional to m_i^2 have been neglected in (2.44). This correction however effects only the overall normalisation of the oscillation formula, not the functional dependence on L arising from the interference term. Integrating over δ_π using (2.21) and (2.22), as well as over t_D, gives the probability of muon decay, into the final state $e^+\nu_k\overline{\nu}_l$, at distance L, from the production point, where all kinematical quantities are expressed in terms of Δm^2, m_π and m_μ:

$$
\begin{aligned}
&P(e^+\nu_k\overline{\nu}_l|L)\\
&= \frac{\pi^{\frac{3}{2}}\Gamma_\mu^2}{3\sqrt{3}} \exp\left[-\frac{8}{3}\left(\frac{\Gamma_\mu m_\mu m_\pi}{m_\pi^2-m_\mu^2}L\right)^2\right]\left[1-\frac{4}{3}\frac{\Gamma_\mu m_\mu m_\pi}{m_\pi^2-m_\mu^2}L\right]^2\\
&\times |\langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle|^2 \exp\left[-\frac{2\Gamma_\mu m_\pi m_\mu(m_\pi^2-m_\mu^2)}{(m_\pi^2+m_\mu^2)^3}L\right]\\
&\times |\langle \nu_0\mu^+|T_R|\pi^+\rangle|^2\\
&\times \left\{1-\sin 2\theta \exp\left[-\frac{2\Gamma_\pi m_\pi^2 m_\mu^2 \Delta m^2}{(m_\pi^2-m_\mu^2)^3}L\right]\right.\\
&\qquad\left.\times \cos\frac{2m_\pi m_\mu^2\Delta m^2}{(m_\pi^2-m_\mu^2)^2}L\right\}. \qquad (2.48)
\end{aligned}
$$

In this expression the correction due to the damping factor of the interference term

$$F^\mu(W_\pi) = \exp[-(\alpha_\pi^\mu(1)-\alpha_\pi^\mu(2))^2\Gamma_\pi^2/12] \qquad (2.49)$$

arising from the integral over the physical pion mass has been neglected. For $\Delta m^2 = (1\,\mathrm{eV})^2$ and $L = 30\,\mathrm{m}$ the numerical value of this factor is $\exp(-5.2\times 10^{-30})$. Denoting by $\phi_{12}^{\mu,\pi}$ the argument of the cosine in (2.48), the exponential damping factor due to the pion lifetime may be written as

$$F^\mu(\Gamma_\pi) = \exp\left(-\frac{\Gamma_\pi m_\pi}{(m_\pi^2-m_\pi^2)}\phi_{12}^{\mu,\pi}\right). \qquad (2.50)$$

For $\phi_{12}^{\mu,\pi} = 1$, $F^\mu(\Gamma_\pi) = \exp[-4.4\times 10^{-16}]$ so, as in the neutrino oscillation case, the pion lifetime damping of the interference term is very small.

Introducing the reduced muon decay amplitude

$$
\begin{aligned}
\langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle &= U_{ek}U_{\mu l}\langle e^+\nu_k\overline{\nu}_l|T_R|\mu^+\rangle\\
&\simeq U_{ek}U_{\mu l}\langle e^+\nu_0\overline{\nu}_0|T_R|\mu^+\rangle, \qquad (2.51)
\end{aligned}
$$

the total muon decay probability is given by the incoherent sum over the four possible final states containing massive neutrinos:

$$
\begin{aligned}
P(e^+\nu\overline{\nu}|L) &= \sum_{k=1}^{2}\sum_{l=1}^{2} P(e^+\nu_k\overline{\nu}_l|L)\\
&= \sum_{k=1}^{2}|U_{ek}|^2\sum_{l=1}^{2}|U_{\mu l}|^2 P(e^+\nu_0\overline{\nu}_0|L)\\
&= P(e^+\nu_0\overline{\nu}_0|L), \qquad (2.52)
\end{aligned}
$$

where the unitarity of the MNS matrix has been used. $P(e^+\nu_0\overline{\nu}_0|L)$ is given by the replacement of $\langle e^+\nu_k\overline{\nu}_l|T|\mu^+\rangle$ by $\langle e^+\nu_0\overline{\nu}_0|T_R|\mu^+\rangle$ in (2.48). Equation (2.52) shows that the muon decay width is independent of the values of the MNS matrix elements.

Corrections due to time dilatation and the Doppler effect are calculated in a similar way to the neutrino oscillation case with the results (see Appendix A)

$$F^\mu(\mathrm{Dop}) = \exp\left\{-\left(\frac{\overline{p}_\pi m_\mu^2\Delta m^2 v_0^\mu}{2m_\pi P_0^3}\left[\frac{3}{2}-\frac{E_0^\mu}{m_\pi}\right]L\right)^2\right\} \qquad (2.53)$$

and

$$\phi^\mu(\mathrm{Dop}) = \frac{3}{2}\left(\frac{\overline{p}_\pi}{m_\pi}\right)^2\frac{m_\mu^2\Delta m^2}{P_0^3}\left[1-\frac{E_0^\mu}{2m_\pi}\right]L. \qquad (2.54)$$

As for neutrino oscillations, the corresponding corrections are very small for oscillation phases of order unity.

The phase of the cosine in the interference term is the same in neutrino and muon oscillations, as can be seen by comparing (2.26) and (2.48). It follows that the target or detector size correction (see (2.34)) is the same in both cases.

Neutrino and muon oscillations from pion decay at rest then have an identical oscillation phase for given values of Δm^2 and L. In view of the much larger event rate that is possible, it is clearly very advantageous in this case to observe muons rather than neutrinos, since the rate of neutrino oscillation events is severely limited by the very small neutrino interaction cross section. In fact, it not necessary to observe muon decay, as in the example discussed above. The oscillation formula applies equally well if the muons are observed[10] at the distance L using any high efficiency detector such as, for example, a scintillation counter. According to (1.2), interference between the path amplitudes must occur if the muon detection device does not discriminate muons recoiling against ν_1 from those recoiling against ν_2.

3 Neutrino oscillations following muon decay or beta-decay at rest

The formula describing "$\overline{\nu}_\mu \to \overline{\nu}_e$ neutrino oscillations"[11] following the decay at rest of a μ^+, $\mu^+ \to$

[10] Note that, in this case, the final state of the path amplitude is that of the detection process by which the muon is recorded. As is the neutrino in neutrino oscillations, the muon itself contributes an unobserved intermediate state in the general path amplitude formula (1.2)

[11] This experiment is also conventionally termed "$\overline{\nu}_e$ appearance". As discussed in more detail in Sect. 5 below, "$\overline{\nu}_e$" and "$\overline{\nu}_\mu$" do not exist, as physical states, if neutrinos are massive and the MNS matrix is non-diagonal. It is still, however, current practice in the literature to use the symbols "ν_e", "ν_μ" and "ν_τ" to refer to massive neutrinos. This is still a useful and meaningful procedure if it is employed only to identify, in

$e^+(\nu_1,\nu_2)(\overline{\nu}_1,\overline{\nu}_2)$ is easily derived from the similar formula for π^+ decay at rest, (2.25). Because the neutrino momentum spectrum is continuous, smearing effects due to the finite muon lifetime may be neglected from the outset. The phase increment associated with the neutrino path is then given by (2.14) with the replacements $P_0 \to P_{\overline{\nu}}$ and $\delta_\pi, \delta_i \to 0$, where $P_{\overline{\nu}}$ is the antineutrino momentum. The phase increment of the decaying muon is given by the same replacements in (2.15) with, in addition, $m_\pi \to m_\mu$ and $\Gamma_\pi \to \Gamma_\mu$. The formula, analogous to (2.26), for the time-averaged probability to detect the process $(\overline{\nu}_1,\overline{\nu}_2)p \to e^+n$ at a distance L from the muon decay point is then

$$P(e^+n,\mu|L) = \frac{|\langle e^+n|T_R|p\overline{\nu}_0\rangle|^2|\langle\nu_0\overline{\nu}_0e^+|T_R|\mu^+\rangle|^2}{\Gamma_\mu}2\sin^2\theta\cos^2\theta \times\left\{1-\exp\left[-\frac{\Gamma_\mu\Delta m^2}{4P_{\overline{\nu}}^2}L\right]\cos\left[\frac{\Delta m^2}{P_{\overline{\nu}}}\left(\frac{m_\mu}{2P_{\overline{\nu}}}-1\right)L\right]\right\}. \quad (3.1)$$

The standard neutrino oscillation formula, hitherto used in the analysis of all experiments, has instead the expression $\Delta m^2 L/(2P_{\overline{\nu}})$ for the argument of the cosine term in (3.1). Denoting my Δm_S^2 the value of Δm^2 obtained using the standard formula, and $\Delta m_{\rm FP}^2$ that obtained using the Feynman path (FP) formula (3.1) then

$$\Delta m_{\rm FP}^2 = \frac{\Delta m_S^2}{\frac{m_\mu}{P_{\overline{\nu}}}-2}. \quad (3.2)$$

For a typical value of $P_{\overline{\nu}}$ of 45 MeV, (3.2) implies that $\Delta m_{\rm FP}^2 \simeq 2.9\Delta m_S^2$. Thus the $\overline{\nu}_\mu$ oscillation signal from μ^+ decays at rest reported by the LNSD Collaboration [14] corresponding to $\Delta m_S^2 \simeq 0.5\,({\rm eV})^2$ for $\sin^2 2\theta \simeq 0.02$ implies $\Delta m_{\rm FP}^2 \simeq 1.5\,({\rm eV})^2$ for a similar mixing angle.

For the case of β-decay:

$$N(A,Z) \to N(A,Z+1)e^-(\overline{\nu}_1,\overline{\nu}_2).$$

m_π in the first line of (2.15) is replaced by E_β, the total energy release in the β-decay process:

$$E_\beta = M_N(A,Z) - M_N(A,Z+1), \quad (3.3)$$

where $M_N(A,Z)$ and $M_N(A,Z+1)$ are the masses of the parent and daughter nuclei. That the phase advance of an unstable state, over a time, Δt, is given by $\exp(-{\rm i}E^*\Delta t)$ where E^* is the excitation energy of the state, is readily shown by the application of time-dependent perturbation theory to the Schrödinger equation [17][12]. A more intuitive derivation of this result has also been given in [18]. In the present case, $E^* = E_\beta$. Omitting the lifetime damping correction, which is about eight orders of magnitude smaller than for pion decay, given a typical β-decay lifetime of a few seconds, the time-averaged probability to detect $\overline{\nu}_1,\overline{\nu}_2$ via the process $(\overline{\nu}_1,\overline{\nu}_2)p \to e^+n$, at distance L from the decay point is given by the formula, derived in a similar way to (2.26) and (3.1),

$$P(e^+n,\beta|L) = \frac{|\langle e^+n|T_R|p\overline{\nu}_0\rangle|^2|\langle e^-\overline{\nu}_0N(A,Z+1)|T_R|N(A,Z)\rangle|^2}{\Gamma_\beta} \times\left\{\sin^4\theta+\cos^4\theta + 2\sin^2\theta\cos^2\theta\cos\left[\frac{\Delta m^2}{P_{\overline{\nu}}}\left(\frac{E_\beta}{2P_{\overline{\nu}}}-1\right)L\right]\right\}, \quad (3.4)$$

where $\Gamma_\beta^{-1} = \tau_\beta$ the lifetime of the unstable nucleus $N(A,Z)$. Until now, all experiments have used the standard expression $\Delta m^2 L/(2P_{\overline{\nu}})$ for the neutrino oscillation phase. The values of Δm^2 found should be scaled by the factor $(E_\beta/P_{\overline{\nu}}-2)^{-1}$, suitably averaged over $P_{\overline{\nu}}$, to obtain the Δm^2 given by the Feynman path formula (3.4).

4 Neutrino and muon oscillations following pion decay in flight

In this section, the decays in flight of a π^+ beam with energy $E_\pi \gg m_\pi$ into $\mu^+(\nu_1,\nu_2)$ are considered. As the analysis of the effects of the physical pion and muon masses has been shown above to give negligible corrections to the L dependence of the oscillation formulae, for the case of decays at rest, such effects will be neglected in this discussion of in-flight decays. The overall structure of the path amplitudes for neutrinos and muons is the same as for decays at rest (see (2.1) and (2.35)). However, for in-flight decays, in order to calculate the interfering paths originating at different and terminating at common space-time points, the two-dimensional spatial geometry of the problem must be properly taken into account.

In Fig. 3 a pion decays at A into the 1 mass eigenstate, the neutrino being emitted at an angle θ_1 in the lab system relative to the pion flight direction. If $m_1 > m_2$ a later pion decay into the 2 mass eigenstate at the angle $\theta_1+\delta$ may give a path such that both eigenstates arrive at the point B at the same time. A neutrino interaction $(\nu_1,\nu_2)n \to e^-p$ occurring at this space-time point will then be sensitive to interference between amplitudes corresponding to the paths AB and ACB. The geometry of the triangle ABC and the condition that the 1 and 2 neutrino mass eigenstates arrive at B at the same time gives the following condition on their velocities:

$$\frac{v_1(\nu)}{v_2(\nu)} = \frac{\sin\theta_2}{\sin\theta_1} - \frac{v_1(\nu)}{v_\pi}\frac{\sin(\theta_2-\theta_1)}{\sin\theta_1}. \quad (4.1)$$

Expanding to first order in the small quantity $\delta = \theta_2 - \theta_1$, rearranging, and neglecting terms of $O(m_i^4)$, gives

$$v_2(\nu) - v_1(\nu) = \frac{\Delta m^2}{2E_\nu^2} = \frac{\delta}{\sin\theta_1}\left[\frac{1-v_\pi\cos\theta_1}{v_\pi}\right], \quad (4.2)$$

a concise manner, the type of charged current interaction by which the neutrinos are produced or detected, i.e. "ν_ℓ" means neutrinos (actually several, with different generation numbers) produced together with the charged lepton $\overline{\ell}$ or detected by observation of the charged lepton ℓ. It should not be forgotten however that only the wavefunctions of the mass eigenstates ν_i occur in the amplitudes of standard model processes.

[12] See also (20) of Chapt. V of [11]

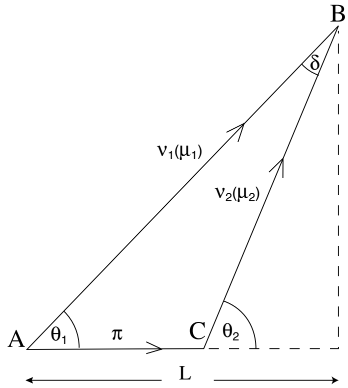

Fig. 3. Two-dimensional spatial geometry for the observation of neutrino or muon oscillations following pion decay in flight. Four possible classical histories of a pion, originally at the point A, are shown. In the first two, the pion decays either into the mass eigenstate $|\nu_1\rangle$, at point A or into $|\nu_2\rangle$ at point C. If $m_1 > m_2$, and for suitable values of the angles θ_1 and θ_2, the neutrinos may arrive at the point B at the same time. If a neutrino detection event, such as $(\nu_1, \nu_2)n \to e^- p$, then occurs at B at this time, the paths AB and ACB will be indistinguishable so that the corresponding amplitudes must be superposed, as in (1.2), to calculate the probability of the overall decay-propagation-detection process. The third and fourth classical histories are similar, except that the neutrino mass eigenstates are replaced by the corresponding recoil muons. The muons in the different histories may arrive at point B, at the same time, leading to interference and "muon oscillations" if they are detected there

where the relation

$$v_i(\nu) = 1 - \frac{m_i^2}{2E_\nu^2} + O(m_i^4) \tag{4.3}$$

has been used. Rearranging (4.2):

$$\delta = \frac{\Delta m^2}{2E_\nu^2}\left[\frac{v_\pi \sin\theta_1}{1 - v_\pi \cos\theta_1}\right]. \tag{4.4}$$

The difference in phase of the neutrino paths AB and CB is (see (2.14))

$$\phi_{12}^\nu = \frac{m_1^2 AB}{P_1} - \frac{m_2^2 CB}{P_2} + O(m_1^4, m_2^4). \tag{4.5}$$

Since the angle δ is $\simeq \Delta m^2$, the difference between AB and CB is of the same order, and so

$$\phi_{12}^\nu = \frac{\Delta m^2 L}{\cos\theta_1 E_\nu} + O(m_1^4, m_2^4), \tag{4.6}$$

where $P_1 \simeq P_2 \simeq E_\nu$, the measured neutrino energy. From the geometry of the triangle ABC:

$$\frac{\mathrm{AC}}{\sin\delta} = \frac{\mathrm{AB}}{\sin\theta_2} = \frac{L}{\cos\theta_1 \sin(\theta_1 + \delta)}. \tag{4.7}$$

So, to first order in δ, and using (4.4)

$$AC \equiv \Delta x_\pi = \frac{L\delta}{\cos\theta_1 \sin\theta_1} = \frac{\Delta m^2 L}{2E_\nu^2 \cos\theta_1}\frac{v_\pi}{(1 - v_\pi\cos\theta_1)} \tag{4.8}$$

and

$$\Delta t_\pi = \frac{\Delta x_\pi}{v_\pi} = \frac{\Delta m^2 L}{2E_\nu^2 \cos\theta_1}\frac{1}{(1 - v_\pi\cos\theta_1)}. \tag{4.9}$$

Equations (4.8) and (4.9) give for the phase increment of the pion path

$$\Delta\phi^\pi = m_\pi(\tau_2 - \tau_1) = m_\pi \Delta\tau = E_\pi \Delta t_\pi - p_\pi \Delta x_\pi = \frac{\Delta m^2 E_\pi L}{2E_\nu^2 \cos\theta_1}\frac{(1 - v_\pi^2)}{(1 - v_\pi\cos\theta_1)}. \tag{4.10}$$

In (4.10), the Lorentz-invariant character of the propagator phase is used. Setting $\cos\theta_1 = 1$ and $v_\pi = 0$ gives for $\Delta\phi^\pi$ a prediction consistent with that obtained from (2.15). Equation (4.6) and (4.10) give for the total phase difference of the paths AB, ACB

$$\phi_{12}^{\nu,\pi} = \Delta\phi^{\mathrm{AB}} - \Delta\phi^{\mathrm{ACB}} = \phi_{12}^\nu - \Delta\phi^\pi = \frac{\Delta m^2 L}{\cos\theta_1 E_\nu}\left[1 - \frac{E_\pi}{2E_\nu}\frac{(1 - v_\pi^2)}{(1 - v_\pi\cos\theta_1)}\right]. \tag{4.11}$$

Using the expressions, valid in the ultra-relativistic (UR) limit where $v_\pi \simeq 1$

$$1 - v_\pi \cos\theta_1 = \frac{m_\pi^2}{E_\pi^2}\frac{1}{(1 + \cos\theta_\nu^*)} \tag{4.12}$$

and

$$E_\nu = \frac{E_\pi(m_\pi^2 - m_\mu^2)}{2m_\pi^2}(1 + \cos\theta_\nu^*), \tag{4.13}$$

where θ_ν^* is the angle between the directions of the pion and neutrino momentum vectors in the pion rest frame, (4.11) may be rewritten as

$$\phi_{12}^{\nu,\pi} = -\frac{\Delta m^2}{\cos\theta_1 E_\nu}\frac{m_\mu^2}{(m_\pi^2 - m_\mu^2)}L. \tag{4.14}$$

For $\theta_1 = 0$ the oscillation phase is the same as for pion decay at rest (see (2.26)) since in the latter case $E_\nu \simeq P_0 = (m_\pi^2 - m_\mu^2)/(2m_\pi)$. Using (4.14), the probability to observe a neutrino interaction, at point B, produced by the decay product of a pion decay occuring within a region of length l_{Dec} ($\ll L$) centered at the point A, in a beam of energy E_π, is given by a formula analogous to (2.26):

$$P(e^- p|L, \theta_1) = \frac{l_{\mathrm{Dec}} m_\pi \Gamma_\pi}{E_\pi}|\langle e^- p|T_R|n\nu_0\rangle|^2 |\langle \nu_0 \mu^+|T_R|\pi^+\rangle|^2 \times \sin^2\theta\cos^2\theta\left\{1 - \cos\frac{m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)E_\nu\cos\theta_1}\right\}. \tag{4.15}$$

As in the case of pion decay at rest, (2.26), the oscillation phase differs by the factor $2m_\mu^2/(m_\pi^2 - m_\mu^2) = 2.685$ from that given by the standard formula.

The derivation of the formula describing muon oscillations following pion decays in flight is very similar to that just given for neutrino oscillations. The condition on the velocities so that the muons recoiling against the different neutrino mass eigenstates arrive at the point B (see Fig. 3) at the same time, is given by a formula analogous to (4.2):

$$\begin{aligned} \Delta v(\mu) &= v_2(\mu) - v_1(\mu) \\ &= \frac{v_1(\mu)[v_1(\mu) - v_\pi \cos\theta_1]}{v_\pi \sin\theta_1}\delta. \end{aligned} \qquad (4.16)$$

The formula relating the neutrino masses to the muon velocities is, however, more difficult to derive than the corresponding relation for neutrinos, (4.3), as the decay muons are not ultra-relativistic in the pion rest frame. The details of this calculation are given in Appendix B. The result is

$$\begin{aligned} \Delta v(\mu) = &\frac{m_\mu^2(m_\pi^2 + m_\mu^2)\Delta m^2}{E_\mu^2(m_\pi^2 - m_\mu^2)^2}\left(1 - \frac{2m_\mu^2 E_\pi}{(m_\pi^2 + m_\mu^2)E_\mu}\right) \\ &+ O(m_1^4, m_2^4). \end{aligned} \qquad (4.17)$$

Using (4.16) and the relation, valid to first order in δ,

$$\Delta t = \frac{L\delta}{v_\pi \cos\theta_1 \sin\theta_1}, \qquad (4.18)$$

where Δt is the flight time of the pion from A to C in Fig. 3 (and also the difference in the times-of-flight of the muons recoiling against the two neutrino eigenstates), the angle δ may be eliminated to yield

$$\Delta t = \frac{\Delta v(\mu) L}{v_1(\mu)\cos\theta_1[v_1(\mu) - v_\pi \cos\theta_1]}. \qquad (4.19)$$

Using now the kinematical relation (see Appendix B):

$$v_1(\mu) - v_\pi \cos\theta_1 = \frac{(m_\pi^2 + m_\mu^2)}{2E_\pi E_\mu}\left(1 - \frac{2m_\mu^2 E_\pi}{(m_\pi^2 + m_\mu^2)E_\mu}\right) \qquad (4.20)$$

and the expression for the phase difference of the paths AB and ACB,

$$\phi_{12}^{\mu,\pi} = \Delta\phi^{\mathrm{AB}} - \Delta\phi^{\mathrm{ACB}} = \Delta t\left(\frac{m_\mu^2}{E_\mu} - \frac{m_\pi^2}{E_\pi}\right), \qquad (4.21)$$

together with (4.19), it is found, taking the UR limit, where $v_1(\mu), v_\pi \simeq 1$, that

$$\phi_{12}^{\mu,\pi} = \frac{2m_\mu^2 \Delta m^2}{E_\mu^2(m_\pi^2 - m_\mu^2)^2}\left[\frac{(m_\mu^2 E_\pi - m_\pi^2 E_\mu)L}{\cos\theta_1}\right]. \qquad (4.22)$$

The probability of detecting a muon decay at B is then

$$\begin{aligned} &P(e^+\nu\overline{\nu}|L,\theta_1) \\ &= \frac{l_{\mathrm{Dec}} m_\pi \Gamma_\pi}{E_\pi}|\langle e^+\nu_0\overline{\nu}_0|T_R|\mu^+\rangle|^2 \\ &\quad \times \exp\left[-\frac{\Gamma_\mu m_\mu}{E_\mu\cos\theta_1}L\right]|\langle\nu_0\mu^+|T_R|\pi^+\rangle|^2 \\ &\quad \times \left\{1 - \sin 2\theta \cos\frac{2m_\mu^2\Delta m^2}{E_\mu^2(m_\pi^2 - m_\mu^2)^2}\right. \\ &\quad \left.\times\left[\frac{(m_\mu^2 E_\pi - m_\pi^2 E_\mu)L}{\cos\theta_1}\right]\right\}, \end{aligned} \qquad (4.23)$$

where l_{Dec} is defined in the same way as in (4.15).

5 Discussion

The quantum mechanics of neutrino oscillations has been surveyed in recent review articles [19–21], where further extensive lists of references may be found.

In this section, the essential difference between the calculations presented in the present paper and all previous treatments in the literature of the quantum mechanics of neutrino oscillations, as cited in the above review articles, will be summarised and applications of the Feynman path amplitude description to other chains of physical processes in space-time will be briefly discussed. A more extended critical review of the existing literature may be found in [25].

Hitherto, it has been assumed that the neutrino source produces a "lepton flavour eigenstate" that is a superposition of mass eigenstates, at some fixed time. In this paper it is, instead, assumed following Shrock [22,23] that the neutrino mass eigenstates are produced incoherently in different physical processes. This follows from the structure of the leptonic charged current in the electroweak standard model:

$$J_\mu(\mathrm{CC})^{\mathrm{lept}} = \sum_{\alpha,i}\overline{\psi}_\alpha\gamma_\mu(1-\gamma_5)U_{\alpha i}\psi_{\nu_i}. \qquad (5.1)$$

Only the wavefunctions of the physical neutrino mass eigenstates ν_i appear in this current, and hence in the initial or final states of any physical process. A consequence is that the neutrino mass eigenstates can be produced at different times in the path amplitudes corresponding to different mass eigenstates. It has recently been shown that experimental measurements of the decay width ratio $\Gamma(\pi \to e\nu)/\Gamma(\pi \to \mu\nu)$ and of the MNS matrix elements are inconsistent with the production of a coherent "lepton flavour eigenstate" in pion decay [24] and that the "equal time" or "equal velocity" hypothesis resulting from this assumption underestimates, by a factor of two, the contribution of neutrino propagation to the oscillation phase [25]. As demonstrated above, allowing for the possibility of different production times of the neutrinos results in an important, decay process dependent, contribution to the oscillation phase from the propagator of the source particle. The non-diagonal elements of $U_{\alpha i}$ in (5.1) describe violation of lepton flavour (or generation number) by the weak charged current. For massless neutrinos, the MNS matrix becomes diagonal; lepton flavour is conserved within each generation, and the familiar "lepton flavour eigenstates" are given by the replacements $\nu_1 \to \nu_e$, $\nu_2 \to \nu_\mu$, $\nu_3 \to \nu_\tau$. Only in this case are the lepton flavour eigenstates physical, being all mass eigenstates of vanishing mass.

The standard derivation of the neutrino oscillation phase will now be considered, following the treatment of [3], but using the notation of the present paper. The calculation is performed assuming an initial "muon flavour eigenstate of the neutrino" that is a superposition of the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$:

$$|\nu_\mu\rangle = U_{\mu 1}|\nu_1, p\rangle + U_{\mu 2}|\nu_2, p\rangle = -\sin\theta|\nu_1, p\rangle + \cos\theta|\nu_2, p\rangle, \tag{5.2}$$

where $|\nu_i, p\rangle$ are mass eigenstates of fixed momentum p. This flavour eigenstate is assumed to evolve with laboratory time, t, according to fixed energy solutions of the non-relativistic Schrödinger equation into the mixed flavour state $|\alpha, t\rangle$:

$$|\alpha, t\rangle = -\sin\theta \mathrm{e}^{-\mathrm{i}E_1 t}|\nu_1, p\rangle + \cos\theta \mathrm{e}^{-\mathrm{i}E_2 t}|\nu_2, p\rangle, \tag{5.3}$$

where E_1, E_2 are the laboratory energies of the neutrino mass eigenstates. The amplitude for transition into the "electron flavour eigenstate":

$$|\nu_e, p\rangle = U_{e1}|\nu_1, p\rangle + U_{e2}|\nu_2, p\rangle = \cos\theta|\nu_1, p\rangle + \sin\theta|\nu_2, p\rangle \tag{5.4}$$

at time t is then, using (5.3) and (5.4)

$$\langle\nu_e, p|\alpha, t\rangle = \sin\theta\cos\theta\left(-\mathrm{e}^{-\mathrm{i}E_1 t} + \mathrm{e}^{-\mathrm{i}E_2 t}\right). \tag{5.5}$$

Because it is assumed that the neutrinos have the same momentum but different energies,

$$E_i = \sqrt{p^2 + m_i^2} = p + \frac{m_i^2}{2p} + O(m_i^4), \tag{5.6}$$

and using (5.5) and (5.6), the probability of the flavour state ν_e at time t is found to be

$$P(\nu_e, t) = |\langle\nu_e, p|\alpha, t\rangle|^2 = 2\cos^2\theta\sin^2\theta\left(1 + \cos\left[\frac{(m_1^2 - m_2^2)}{2p}t\right]\right). \tag{5.7}$$

Finally, since the velocity difference of the neutrino mass eigenstates is $\mathrm{O}(\Delta m^2)$, then, to the same order in the oscillation phase, the replacement $t \to L$ can be made in (5.7) to yield the standard oscillation phase of (1.1).

The following comments may be made on this derivation.

(i) The time evolution of the neutrino mass eigenstates in (5.3) according to the Schrödinger equation yields a non-Lorentz-invariant phase $\simeq Et$, to be compared with the Lorentz-invariant phase $\simeq m^2 t/E$ given in (2.14) above. Although the two expressions agree in the non-relativistic limit $E \simeq m$ it is clearly inappropriate to use this limit for the description of neutrino oscillation experiments. It may be noted that the Lorentz-invariant phase is robust relative to different kinematical approximations. The same result is obtained to order m^2 for the phase of spatial oscillations independent of whether the neutrinos are assumed to have equal momenta or energies. This is not true in the non-relativistic limit. Assuming equal momenta gives the standard result of (1.1), whereas the equal energy hypothesis results in a vanishing oscillation phase. A contrast may be noted here with the standard treatment of neutral kaon oscillations, which follows closely the derivation in (5.2) to (5.7) above, except that the particle phases are assumed to evolve with time according to the Lorentz-invariant expression, $\exp[-\mathrm{i}m\tau]$, where m is the particle mass and τ is its proper time, in agreement with (2.11).

(ii) As pointed out in [6], the different neutrino mass eigenstates do not have equal momenta as assumed in (5.2) and (5.6). The approximation of assuming equal momenta might be justified if the fractional change in the momentum of the neutrino due to a non-vanishing mass were much less than that of the energy. However, in the case of pion decay as is readily shown from (2.4) and (2.6) above, the ratio of the fractional shift in momentum to that in energy is actually $(m_\pi^2 + m_\mu^2)/(m_\pi^2 - m_\mu^2) = 3.67$; so, in fact, the opposite is the case.

(iii) The derivation of (5.7) is carried out in the abstract Hilbert space of the neutrino mass eigenstates or "lepton flavour eigenstates" without any reference to the production or detection processes necessary for the complete description of an experiment in which "neutrino oscillations" may be observed. In this calculation the "mass" and "flavour" bases are treated as physically equivalent. However in standard model amplitudes only states of the mass basis appear. Also it has been pointed out that "flavour momentum eigenstates" cannot be defined in a theoretically consistent manner [26]. Their existence is, in any case, excluded by experiment for the case of pion decay [24].

(iv) What are the physical meanings of t, p in (5.3)? In this equation it is assumed that the neutrino mass eigenstates are both produced, and both detected, at the same times. Thus both have the same time-of-flight t. The momentum p cannot be the same for both eigenstates, as assumed in (5.6), if both energy and momentum are conserved in the decay process. For any given value of the laboratory time t the different neutrino mass eigenstates must be at different space-time positions because they have different velocities[13], if it is assumed that both mass eigenstates are produced at the same time. It then follows that the different mass eigenstates cannot be probed, at the same space-time point, by a neutrino interaction, whereas the latter must occur at a definite space-time point in every detection event. In fact, there is an inconsistent treatment of the velocity of the neu-

[13] This is true not only in the case of energy-momentum conservation, but also if it is assumed, as in the derivation of the standard formula, that the neutrinos have the same momentum but different energies.

trinos. Equal production times imply equal space-time velocities, whereas it is assumed that "kinematical velocities" defined as p_i/E_i are different for the different mass eigenstates.

(v) The historical development of the calculation of the neutrino oscillation phase is of some interest. The first published prediction [27] actually obtained a phase a factor two larger than (1.1) i.e. in agreement with the contribution from neutrino propagation found in the present paper. This prediction was later used, for example, in [28]. The derivation sketched above, leading to the standard result of (1.1) was later given in [29]. A subsequent paper [30] by the authors of [28], published shortly afterwards, cited both [27] and [29], but used now the prediction of the latter paper. No comment was made on the factor of two difference in the two calculations. In a later review article, [31], by the authors of [28] a calculation similiar to that of [29] was presented in detail. Subsequently, all neutrino oscillation experiments have been analysed on the assumption of the standard oscillation phase of (1.1).

It may be thought that the kinematical and geometrical inconsistencies mentioned in points (ii) and (iv) above result from a too classical approach to the problem. After all, what does it mean, in quantum mechanics, to talk about the "position" and "velocity" of a particle, in view of the Heisenberg uncertainty relations [32]? Following the original suggestion of [33] in almost all subsequent work on the quantum mechanics of neutrino oscillations, a "wave packet" description of the neutrino mass eigenstates has been cojectured. In this approach, both the "source" and also possibly the "detector" in the neutrino oscillation experiment are described by coherent spatial wave packets. An extended critical discussion of this approach may be found in [25]. There it is concluded that the introduction of wave packets in this ad hoc manner is without physical foundation.

Correlated production and detection of neutrinos and muons produced in pion decay were considered in [34,35]. The introduction to [35] contains a valuable discussion of the universality of the "particle oscillation" phenomenon. It is pointed out that this is a consequence of the general principle of amplitude superposition in quantum mechanics, and so is not a special property of the K^0–$\overline{K}^0$, B^0–$\overline{B}^0$ and neutrino systems which are usually discussed in this context. This paper used a covariant formalism that employed the "energy representation" of the space-time propagator. Correlated spatial oscillations of neutrinos and muons are predicted, though with interference phases different from the results of both the present paper and the standard formula. Pion and muon lifetime effects were mentioned in [35], but neither the role of the pion lifetime in enabling different propagation times for the neutrinos nor the momentum smearing, induced by the Fourier-transform-related Breit–Wigner amplitudes, were discussed.

The claim of [34] that correlated neutrino–muon oscillations should be observable in pion decay was questioned in [36]. The authors of the latter paper attempted to draw conclusions on the possibility, or otherwise, of particle oscillations by using "plane waves", i.e. energy-momentum eigenfunctions. As is well known, such wavefunctions are not square integrable, and so can yield no spatial information. The probability to find a particle described by such a wave function in any finite spatial volume is zero. Due to the omission of the (infinite) normalisation constants of the wavefunctions many of the equations in [36] are, as previously pointed out [37], dimensionally incorrect. Momentum wavepackets for the decaying pion were also discussed in [36]. Although exact energy-momentum conservation constraints were used, it was assumed, as in [34,35], that the muons and the different neutrino mass eigenstates are both produced and detected at common points ((35) of [36]). The latter assumption implies equal velocities, yielding the standard neutrino oscillation phase. The authors of [36] drew the following conclusions:

(a) Correlated μ–ν oscillations of the type discussed in [35] could be observed, though with different oscillation phases.
(b) Oscillations would not be observed if only the muon is detected.
(c) Neutrino oscillations can be observed even if the muon is not detected.

Conclusion (b) is a correct consequence of the (incorrect) assumption that the muons recoiling against the different neutrino mass eigenstates have the same velocity. As both muons have the same mass they will have equal proper time increments. So according to (2.12) the phase increments will also be equal and the interference term will vanish. The conclusion (c) is in agreement with the prediction of (3.22) of [35]. The path amplitude calculation of the present paper shows that conclusion (b) is no longer valid when the different possible times of propagation of the recoiling muons are taken into account.

It is clearly of great interest to apply the calculational method developed in the present paper to the case of neutral kaon and b-meson oscillations. Indeed the use of the invariant path amplitude formalism has previously been recommended [38] for experiments involving correlated pairs of neutral kaons. Here, just a few remarks will be made on the main differences to be expected from the case of neutrino or muon oscillations. A further discussion can be found in [25].

In the case of neutrino and muon oscillations, the interference effect is possible as the different neutrino eigenstates can be produced at different times. This is because the decay lifetimes of all interesting sources (pions, muons, β-decaying nuclei) are much longer than the time difference between the paths corresponding to the interfering amplitudes. To see if a similar situation holds in the case of K_S–K_L oscillations, three specific examples will be considered with widely differing momenta of the neutral kaons:

(I) $\phi \to K_S K_L$;
(II) $\pi^- p \to \Lambda K^0$ at $\sqrt{s} = 2\,\text{GeV}$;
(III) $\pi^- p \to \Lambda K^0$ at $\sqrt{s} = 10\,\text{GeV}$.

These correspond to neutral kaon centre-of-mass momenta of 108 MeV, 750 MeV and 5 GeV respectively. In each case the time difference (Δt_K) of production of K_{S} and K_{L} mesons, in order that they arrive at the same time at a point distant $c\gamma_K\tau_S$ (where γ_K is the usual relativistic parameter) from the source in the centre-of-mass frame is calculated. Exact relativistic kinematics is assumed and only leading terms in the mass difference $\Delta m_K = m_L - m_S$ are retained. Taking the value of Δm_K and the various particle masses from [40] the following results are found for Δt_K in the three cases: (I) 2.93×10^{-24} s, (II) 8.3×10^{-25} s and (III) 6.4×10^{-26} s. For comparison, for neutrino oscillations following pion decay at rest, with $\Delta m^2 = (1\,\mathrm{eV})^2$ and $L = 30\,\mathrm{m}$, (2.29) gives $\Delta t_\nu = 5.6\times10^{-23}$ s. The result (I) may be compared with the mean life of the ϕ meson of 1.5×10^{-22} s [40]. Thus the ϕ lifetime is a factor of about 27 larger than Δt_K indicating that K_{S}–K_{L} interference should be possible by a similar mechanism to neutrino oscillations following pion decays, i.e. without invoking velocity smearing of the neutral kaon mass eigenstates. In cases (I) and (II) the interference effects observed will depend on the "characteristic time" of the non-resonant (and hence incoherent) strong interaction process, a quantity that has, hitherto, not been susceptible to experimental investigation[14]. If this time is much less than, or comparable to, Δt_K, essentially equal velocities (and therefore appreciable velocity smearing) of the eigenstates will be necessary for interference to occur. Since Δm_K and Γ_S are comparable in size, velocity smearing effects are expected to be, in any case, much larger than for neutrino oscillations following pion decay. These effects may be roughly estimated by using the Gaussian approximation (2.21) of the present paper. The main contribution to the velocity smearing is due to the variation of the physical mass of the K_{S} rather than those of the K_{L} or Λ.

Taking into account this smearing of the physical mass of the K_{S}, the change in velocity as compared to that corresponding to the pole mass, due to this effect, is of the same order as that due to the K_{L}–K_{S} mass difference. The K_{S} and K_{L} can then have almost equal velocities and hence almost equal production times, so that

$$\Delta\tau_L \simeq \Delta\tau_S \simeq \Delta\tau. \tag{5.8}$$

Using the formula $\Delta\tau = m_{K^0}L/p_{K^0}$, relating the decay distance, D, to the proper time increment $\Delta\tau$, and (2.11) gives for the phase increments corresponding to K_{L}, K_{S}:

$$\Delta\phi_i = \frac{m_i m_{K^0} D}{p_{K^0}} \quad (i = L, S). \tag{5.9}$$

Setting $m_{K^0} = (m_L + m_S)/2$ in (5.9) yields, for the oscillation phase

$$\phi_{LS} \simeq \frac{\Delta m_K m_{K^0} D}{p_{K^0}} = \frac{(m_L - m_S)(m_L + m_S)D}{2p_{K^0}} = \frac{(m_L^2 - m_S^2)D}{2p_{K^0}}. \tag{5.10}$$

[14] A similar physical quantity has been considered in [41], where the possibility of observable modifications to the exponential decay law and the Breit–Wigner line shape distribution is suggested.

This is the same as the standard formula for neutrino oscillations, also derived, as discussed in detail in Ref [25], on the assumption of equal velocities for different neutrino mass eigenstates. The conventionally used formula for neutral kaon oscillations is the first member of (5.10) i.e. equal proper time intervals, as calculated from the observed decay distance, and hence equal velocities, are assumed. Unlike for the case of neutrino oscillations where this assumption is definitely incorrect [25], it is likely to be a good first approximation in the case of neutral kaon (and neutral b-meson) oscillations. Although it would be interesting to estimate quantitatively the effect of velocity smearing and contributions to the oscillation phase due to different production times, such corrections are not expected to be large, so that the experimental values of Δm_K and Δm_B should be little affected.

It is interesting to note that the possibility of "Λ oscillations" in the processes $\pi^- p \to \Lambda(K_{\mathrm{L}}^0, K_{\mathrm{S}}^0)$ analogous to "μ oscillations" in the processes $\pi \to \mu(\nu_1, \nu_2)$ has previously been proposed in the literature [39].

For the B_1–B_2 oscillation case, analogous to (I) above, $\Upsilon(4S) \to B_1 B_2$, where $p_B = 335\,\mathrm{MeV}$, the value of Δt_B is found to be 1.4×10^{-22} s, to be compared with $\tau(\Upsilon(4S)) = 4.7\times10^{-23}$ s [40], which is a factor 3 smaller. Thus, velocity smearing effects are expected to play an important role in B_1–B_2 oscillations. This is possible, since the neutral b-meson decay width (4.3×10^{-10} MeV) and the mass difference (3.1×10^{-10} MeV) have similar sizes.

In closing, it is interesting to mention two types of atomic physics experiments where interference effects similar to the conjectured (and perhaps observed [14,42,43]) neutrino oscillations have already been clearly seen.

The first is quantum beat spectroscopy [44]. This type of experiment, which has previously been discussed in connection with neutrino oscillations [37], corresponds closely to the gedanken experiment used by Heisenberg [10] to exemplify the fundamental law of quantum mechanics, (1.2). The atoms of an atomic beam are excited by passage through a thin foil or a laser beam. The quantum phase of an atom with excitation energy E^* evolves with time according to $\exp(-\mathrm{i}E^*\Delta t)$ (see the discussion after (3.3) above). If decay photons from two nearby states with excitation energies E_α^* and E_β^* are detected after a time interval Δt (for example by placing a photon detector beside the beam at a variable distance d from the excitation foil) a cosine interference term with phase:

$$\phi_{\mathrm{beat}} = \frac{(E_\alpha^* - E_\beta^*)d}{\overline{v}_{\mathrm{atom}}}, \tag{5.11}$$

where $\overline{v}_{\mathrm{atom}}$ is the average velocity of the atoms in the beam, is observed [44]. An atom in the beam, before excitation, corresponds to the neutrino source pion. The excitation process corresponds to the decay of the pion. The propagation of the two different excited states, *alternative* histories of the initial atom, correspond to the *alternative* propagation of the two neutrino mass eigenstates. Finally, the decay of the atoms and the detection of a *single* photon corresponds to the neutrino detection process. The particular importance of this experiment for the path amplitude calculations presented in the present paper is that it demonstrates, experimentally, the important contribution to the interference phase of the space-time propagators of excited atoms, in direct analogy to the similar contributions of unstable pions, muons and nuclei discussed above.

An even closer analogy to neutrino osillations following pion decay is provided by the recently observed process of photodetachment of an electron by laser excitation: the "Photodetachment Microscope" [45]. A laser photon ejects the electron from, for example, an $^{16}O^-$ ion in a beam. The photodetached electron is emitted in an S-wave (isotropically) and with a fixed initial energy. It then moves in a constant, vertical, electric field that is perpendicular to the direction of the ion beam and almost parallel to the laser beam. An upward moving electron that is decelerated by the field eventually undergoes "reflection" before being accelerated towards a planar position-sensitive electron detector situated below the beam and perpendicular to the electric field direction (see Fig. 1 of [45]). In these circumstances, it can be shown [46] that just two classical electron trajectories link the production point to any point in the kinematically allowed region of the detection plane. Typical parameters for $^{16}O^-$ are [47]: initial electron kinetic energy, 102 μeV: detector distance, 51.4 cm; average time-of-flight, 117 ns; difference in emission times for the electrons to arrive in spatial-temporal coincidence at the detector plane, 160 ps. An interference pattern is generated by the phase difference between the amplitudes corresponding to the two allowed trajectories. The phase difference, derived by performing the Feynman path integral of the classical action along the classical trajectories [47], gives a very good description of the observed interference pattern. The extremely close analogy between this experiment and the neutrino oscillation experiments described in Sects. 2 and 3 above is evident. Notice that the neutrinos, like the electrons in the photodetachment experiment, must be emitted at different times, in the alternative paths, for interference to be possible. This is the crucial point that was not understood in all previous treatments of the quantum mechanics of neutrino oscillations.

Actually, [47] contains, in Sect. IV, a path amplitude calculation for electrons in free space that is geometrically identical to the discussion of pion decays in flight presented in Sect. 4 above (compare Fig. 3 of the present paper with Fig. 3 of [47]) The conclusion of [47] is that, in this case, no interference effects are possible for electrons that are mono-energetic in the source rest frame. As is shown in Sect. 4 above, if these electrons are replaced either by neutrinos of different masses from pion decay, or muons recoiling against such neutrinos, observable interference effects are indeed to be expected.

Appendix A

Random thermal motion of the decaying pion in the target has two distinct physical effects on the phase of neutrino oscillations,

$$\phi_{12}^{\nu,\pi}(0) = -\frac{\Delta m^2 L}{P_0} + \frac{m_\pi \Delta m^2 L}{2P_0^2} \tag{A1}$$

(where the first and second terms in (A1) give the contributions of the neutrino and pion paths respectively).

(1) The observed neutrino momentum, P_ν, is no longer equal to P_0, due to the boost from the pion rest frame to the laboratory system (Doppler effect or Lorentz boost).
(2) The time increment of the pion path $t_D - t_0$ (see (2.16)) no longer corresponds to the pion proper time (relativistic time dilatation).

Taking into account (1) and (2) gives for the neutrino oscillation phase

$$\phi_{12}^{\nu,\pi}(\mathrm{corr}) = -\frac{\Delta m^2 L}{P_\nu} + \frac{m_\pi \Delta m^2 L}{2\gamma_\pi P_\nu^2}, \tag{A2}$$

where

$$P_\nu = \gamma_\pi P_0(1 + v_\pi \cos\theta_\nu^*) \quad \text{and} \quad \gamma_\pi = \frac{E_\pi}{m_\pi}.$$

Here, θ_ν^* is the angle between the neutrino momentum vector and the pion flight direction in the pion rest frame. Developing γ_π and v_π in terms of the small quantity p_π/m_π, (A2) may be written as

$$\phi_{12}^{\nu,\pi}(\mathrm{corr}) = \phi_{12}^{\nu,\pi}(0) + \frac{p_\pi}{m_\pi}\frac{\Delta m^2 L}{P_0}\left[1 - \frac{m_\pi}{P_0}\right]\cos\theta_\nu^* + \left(\frac{p_\pi}{m_\pi}\right)^2 \frac{\Delta m^2 L}{2P_0}\left[1 - \frac{3m_\pi}{2P_0}\right]. \tag{A3}$$

Performing now the average of the interference term over the isotropic distribution in $\cos\theta_\nu^*$:

$$\begin{aligned}\langle \cos\phi_{12}^{\nu,\pi}(\mathrm{corr})\rangle_{\theta_\nu^*} &= \frac{1}{4}\mathrm{Re}\int_{-1}^{1} \exp[\mathrm{i}\phi_{12}^{\nu,\pi}(\mathrm{corr})]\mathrm{d}\cos\theta_\nu^* \\ &= \frac{1}{2}\mathrm{Re}\exp\left\{\mathrm{i}\phi_{12}^{\nu,\pi}(0) + \left(\frac{p_\pi}{m_\pi}\right)^2\left(\mathrm{i}\frac{\Delta m^2 L}{2P_0}\left[1 - \frac{3m_\pi}{2P_0}\right] - \frac{1}{6}\left(\frac{\Delta m^2 L}{P_0}\left[1 - \frac{m_\pi}{P_0}\right]\right)^2\right)\right\}. \end{aligned} \tag{A4}$$

In deriving (A4) the following approximate formula is used:

$$\frac{1}{2}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\alpha c}\mathrm{d}c = \frac{1}{2\mathrm{i}\alpha}\left[\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha}\right] = \frac{\sin\alpha}{\alpha} \simeq 1 - \frac{\alpha^2}{6}, \tag{A5}$$

where

$$\alpha \equiv \frac{p_\pi}{m_\pi}\frac{\Delta m^2 L}{P_0}\left[1-\frac{m_\pi}{P_0}\right] \ll 1.$$

The average over the Maxwell–Boltzmann distribution (2.31) is readily performed by "completing the square" in the exponential, with the result

$$\begin{aligned}&\langle\cos\phi_{12}^{\nu,\pi}(\mathrm{corr})\rangle_{\theta_\nu^*,p_\pi}\\ &= \frac{1}{2}\mathrm{Re}\exp\left\{-\left(\frac{\overline{p}_\pi \Delta m^2 L}{2m_\pi P_0}\left[1-\frac{m_\pi}{P_0}\right]\right)^2\right.\\ &\quad\left.+\,\mathrm{i}\left[\phi_{12}^{\nu,\pi}(0)+\frac{3}{4}\left(\frac{\overline{p}_\pi}{m_\pi}\right)^2\left(\frac{\Delta m^2 L}{P_0}\left[\frac{3m_\pi}{2P_0}-1\right]\right)\right]\right\}\\ &\equiv F^\nu(\mathrm{Dop})\cos[\phi_{12}^{\nu,\pi}(0)+\phi^\nu(\mathrm{Dop})], \qquad \text{(A6)}\end{aligned}$$

leading to (2.32) and (2.33) for the Doppler damping factor $F^\nu(\mathrm{Dop})$ and phase shift $\phi^\nu(\mathrm{Dop})$, respectively.

The correction for the effect of thermal motion in the case of muon oscillations $\phi^\nu(\mathrm{Dop})$, respectively, is performed in a similar way. The oscillation phase

$$\phi_{12}^{\mu,\pi}(0) = -\frac{m_\mu^2 E_0^\mu \Delta m^2 L}{2m_\pi P_0^3}+\frac{m_\mu \Delta m^2 L}{2P_0^3} \qquad \text{(A7)}$$

is modified by the Lorentz boost of the muon momentum and energy, and the relativistic time dilatation of the phase increment of the pion path, to

$$\phi_{12}^{\mu,\pi}(\mathrm{corr}) = -\frac{m_\mu^2 E_\mu \Delta m^2 L}{2m_\pi P_\mu^3}+\frac{m_\mu \Delta m^2 L}{2\gamma_\pi P_\mu^3}, \qquad \text{(A8)}$$

where

$$E_\mu = \gamma_\pi E_0^\mu(1+v_\pi v_0^\mu\cos\theta_\mu^*),$$

and v_0^μ is given by (2.37). Developing, as above, in terms of p_π/m_π, gives

$$\begin{aligned}&\phi_{12}^{\mu,\pi}(\mathrm{corr})\\ &= \phi_{12}^{\mu,\pi}(0)+\frac{p_\pi}{m_\pi}\frac{v_0^\mu m_\mu^2 \Delta m^2 L}{P_0^3}\left[\frac{E_0^\mu}{m_\pi}-\frac{3}{2}\right]\cos\theta_\mu^*\\ &\quad+\left(\frac{p_\pi}{m_\pi}\right)^2\frac{m_\mu^2\Delta m^2 L}{P_0^3}\left[\frac{E_0^\mu}{2m_\pi}-1\right]. \qquad \text{(A9)}\end{aligned}$$

Performing the averages over θ_μ^* and p_π then leads to (2.53) and (2.54) for the damping factor $F^\mu(\mathrm{Dop})$ and phase shift $\phi^\mu(\mathrm{Dop})$, respectively.

The effect of the finite longitudinal dimensions of the target or detector is calculated by an appropriate weighting of the interference term according to the value of the distance $X = x_f - x_i$ between the decay and detection points (see Fig. 1). Writing the interference phase as $\phi_{12} = \beta X$, and assuming a uniform distribution of decay points within the target of thickness ℓ_T:

$$\begin{aligned}\langle\cos\phi_{12}\rangle &= \frac{1}{\ell_T}\int_{L-\frac{\ell_T}{2}}^{L+\frac{\ell_T}{2}}\cos\beta X\,\mathrm{d}X\\ &= \frac{2}{\beta\ell_T}\sin\frac{\beta\ell_T}{2}\cos\beta L\\ &\equiv F_{\mathrm{Targ}}\cos\beta L. \qquad \text{(A10)}\end{aligned}$$

Substituting the value of β appropriate to neutrino oscillations yields (2.34). Since the value of β is the same for neutrino and muon oscillations, the same formula is also valid in the latter case. The same correction factor, with the replacement $\ell_T \to \ell_D$ describes the effect of a finite detection region of length ℓ_D:

$$L-\frac{\ell_D}{2}+x_i < x_f < L+\frac{\ell_D}{2}+x_i.$$

Appendix B

The first step in the derivation of (4.17) relating $\Delta v(\mu)$ to Δm^2 is to calculate the angle δ^*, in the centre-of-mass (CM) system of the decaying pion, corresponding to δ in the laboratory (LAB) system (see Fig. 3). It is assumed, throughout, that the pion and muon are ultra-relativistic in the latter system, so that $v_\pi, v_i(\mu) \simeq 1$. The Lorentz transformation relating the CM and LAB systems gives the relation

$$\sin\theta_i = \frac{v_i^*(\mu)\sin\theta_i^*}{\gamma_\pi(1+v_i^*(\mu)\cos\theta_i^*)}, \quad i=1,2. \qquad \text{(B1)}$$

The starred quantities refer to the pion CM system. Making the substitutions $\theta_2 = \theta_1+\delta$, $\theta_2^* = \theta_1^*+\delta^*$, (B1) may be solved to obtain, up to first order in δ, δ^* and Δm^2:

$$\begin{aligned}&\Delta v^*(\mu) = v_2^*(\mu)-v_1^*(\mu)\\ &= \frac{\gamma_\pi(1+v_0^*(\mu)\cos\theta_1^*)^2\delta - v_0^*(\mu)(\cos\theta_1^*+v_0^*(\mu))\delta^*}{\sin\theta_1^*},\end{aligned} \qquad \text{(B2)}$$

where (c.f. (2.37))

$$v_0^*(\mu) = \frac{m_\pi^2-m_\mu^2}{m_\pi^2+m_\mu^2}. \qquad \text{(B3)}$$

Using (2.36) $\Delta v^*(\mu)$ may be expressed in terms of the neutrino mass difference:

$$\Delta v^*(\mu) = \frac{4m_\mu^2 m_\pi^2 \Delta m^2}{(m_\pi^2-m_\mu^2)(m_\pi^2+m_\mu^2)^2}. \qquad \text{(B4)}$$

Eliminating now $\Delta v^*(\mu)$ between (B2) and (B4) gives a relation between δ, δ^* and Δm^2:

$$\begin{aligned}\delta^* &= \frac{\gamma_\pi(1+v_0^*(\mu)\cos\theta_1^*)^2\delta}{v_0^*(\mu)(\cos\theta_1^*+v_0^*(\mu))}\\ &\quad-\frac{4m_\mu^2 m_\pi^2\Delta m^2\sin\theta_1^*}{(m_\pi^2-m_\mu^2)^2(m_\pi^2+m_\mu^2)(\cos\theta_1^*+v_0^*(\mu))}. \qquad \text{(B5)}\end{aligned}$$

In the LAB system, and in the UR limit, the difference of the velocities of the muons recoiling against the two neutrino mass eigenstates is

$$\begin{aligned}\Delta v(\mu) = v_2(\mu)-v_1(\mu) &= \frac{P_2(\mu)}{E_2(\mu)}-\frac{P_1(\mu)}{E_1(\mu)}\\ &\simeq \frac{m_\mu^2}{E_\mu^3}[E_2(\mu)-E_1(\mu)], \qquad \text{(B6)}\end{aligned}$$

where E_μ is the muon energy in the LAB system for vanishing neutrino masses. Making the Lorentz transformation of the muon energy from the pion CM to the LAB frames, and using (2.4) and (2.36) to retain only terms linear in Δm^2 and δ^*, enables (B6) to be re-written as

$$\Delta v(\mu) = \frac{E_\pi}{2E_\mu^3}\left(\frac{m_\pi}{m_\mu}\right)^2 \times \left[\frac{\Delta m^2(\cos\theta_1^* + v_0^*(\mu))}{v_0^*(\mu)} - \delta^*(m_\pi^2 - m_\mu^2)\sin\theta_1^*\right], \quad \text{(B7)}$$

where E_π is the energy of the pion beam. By combining the geometrical constraint equation for the muon velocities, (4.16) with (B5) and (B7) the angles δ and δ^* may be eliminated to yield the equation for the LAB frame velocity difference:

$$\Delta v(\mu) = \frac{E_\pi \Delta m^2}{2m_\pi^2(m_\pi^2 - m_\mu^2)}\frac{A}{B}, \quad \text{(B8)}$$

where

$$A = (v_1(\mu) - v_\pi\cos\theta_1) \times \left\{(\cos\theta_1^* + v_0^*(\mu))^2 + \frac{4m_\mu^2 m_\pi^2 \sin^2\theta_1^*}{(m_\pi^2 + m_\mu^2)^2}\right\} \quad \text{(B9)}$$

$$B = \frac{E_\pi(m_\pi^4 - m_\mu^4)(1 + v_0^*(\mu)\cos\theta_1^*)}{8m_\pi^4 m_\mu^2} \times \left\{\frac{E_\pi^2(m_\pi^2 + m_\mu^2)(1 + v_0^*(\mu)\cos\theta_1^*)^2}{m_\pi^2(m_\pi^2 - m_\mu^2)} \times (\cos\theta_1^* + v_0^*(\mu))(v_1(\mu) - v_\pi\cos\theta_1) + \frac{4m_\mu^2 m_\pi^2\sin^2\theta_1^*}{(m_\pi^2 + m_\mu^2)^2}\right\}. \quad \text{(B10)}$$

To simplify (B8), the quantity $(v_1(\mu) - v_\pi\cos\theta_1)$ is now expressed in terms of kinematic quantities in the pion CM system. Within the UR approximation used,

$$\theta_1, m_\pi/E_\pi, m_\mu/E_\mu \ll 1,$$

so that

$$v_1(\mu) - v_\pi\cos\theta_1 = \frac{1}{2}\left(\frac{m_\pi^2}{E_\pi^2} - \frac{m_\mu^2}{E_\mu^2} + \theta_1^2\right) + O\left(\left(\frac{m_\pi}{E_\pi}\right)^4, \left(\frac{m_\mu}{E_\mu}\right)^4, \theta_1^4\right) \quad \text{(B11)}$$

Writing (B1) to first order in θ_1, and neglecting terms of $O(\theta_1 m_i^2)$:

$$\theta_1 = \frac{m_\pi v_0^* \sin\theta_1^*}{E_\pi(1 + v_0^*(\mu)\cos\theta_1^*)}. \quad \text{(B12)}$$

Using (B12), and expressing E_μ in terms of pion CM quantities, (B11) may be written as

$$v_1(\mu) - v_\pi\cos\theta_1 = \frac{m_\pi^2(m_\pi^2 - m_\mu^2)(\cos\theta_1^* + v_0^*(\mu))}{E_\pi^2(m_\pi^2 + m_\mu^2)(1 + v_0^*(\mu)\cos\theta_1^*)^2}. \quad \text{(B13)}$$

Expressing the RHS of (B13) in terms of E_π and E_μ and using the relation

$$\cos\theta_1^* = \frac{m_\pi^2(2E_\mu - E_\pi) - m_\mu^2 E_\pi}{E_\pi(m_\pi^2 - m_\mu^2)} \quad \text{(B14)}$$

gives (4.20) of the text.

On substituting (B13) into the RHS of (B10), it can be seen that the factor in the large curly brackets is the same in (B9) and (B10), and so cancels in the ratio A/B in (B8). It follows that

$$\Delta v(\mu) = \frac{m_\mu^2 \Delta m^2}{E_\mu^2(m_\pi^2 - m_\mu^2)}\left(\frac{\cos\theta_1^* + v_0^*(\mu)}{1 + v_0^*(\mu)\cos\theta_1^*}\right). \quad \text{(B15)}$$

Finally, using (B3) and (B14) to express the factor in large brackets in (B15) in terms of E_μ and E_π, (4.17) of the text is obtained.

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